EXACT TRIANGLE FOR FIBERED DEHN TWISTS

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ABSTRACT. We use quilted Floer theory to generalize Seidel's long exact sequence in symplectic Floer theory to fibered Dehn twists. We then apply the sequence to construct versions of the Floer and Khovanov-Rozansky exact triangles in Lagrangian Floer theory of moduli spaces of bundles.

CONTENTS

1. INTRODUCTION

Seidel's long exact sequence [42, Theorem 1] describes the effect of a symplectic Dehn twist on Lagrangian Floer cohomology. In many examples (moduli spaces of bundles, nilpotent slices, etc.) the relevant fibrations have Morse-Bott rather than Morse singularities and the associated monodromy maps are *fibered* Dehn twists. Many years ago Seidel suggested that this sequence should generalize to the fibered case. In this paper we show how to carry out this suggestion using quilted Floer theory developed in Wehrheim-Woodward [52], [46], [47]. Quilted Floer theory gives an expression of the third term in the exact triangle as a push-pull functor, similar to the situation in the analogous triangle in algebraic geometry developed by Horja [17].

To state the main result suppose that M is an exact or monotone symplectic manifold. If M has boundary ∂M then we assume that the boundary ∂M is convex so that our spaces of pseudoholomorphic curves satisfy good compactness properties. A *Lagrangian brane* in M is a compact, oriented Lagrangian submanifold L equipped with a grading in the sense of [47] and relative spin structure in the sense of [50]. We say that a Lagrangian brane L is *admissible* if L is monotone in the sense of [47], the

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image of the fundamental group $\pi_1(L)$ of L in $\pi_1(M)$ is torsion for any choice of base point and L has minimal Maslov number at least three. This notion of admissibility is chosen so that any pair L^0, L^1 of admissible Lagrangian branes in M is monotone as a pair, which implies an energy-index relation for pseudoholomorphic strips. This relation in turn implies that disk bubbles cannot obstruct the proof of $\partial^2 = 0$ in the construction of the *Lagrangian Floer homology group* $HF(L^0, L^1)$. The underlying complex for this group is generated by perturbed intersection points of L^0, L^1. The differential counts finite energy holomorphic strips with boundary in L^0, L^1. Taking the Floer groups as morphism spaces, one obtains the cohomology of the Fukaya category of Lagrangian branes in M. For the moment, we work with $\mathbb{Z}_2$ coefficients, although the main result will be stated with $\mathbb{Z}$ coefficients.

To recall Seidel's exact triangle in Lagrangian Floer cohomology, let $C \subset M$ be a Lagrangian sphere equipped with an identification with a unit sphere, and let $\iota_C : C \to C$, $v \mapsto -v$ denote the antipodal map. Associated to C is a *symplectic Dehn twist*

$$\tau_C : M \to M, \quad \tau_C|C = \iota_C, \quad \mathrm{supp}(\tau_C) \subset U_C$$

equal to the antipodal map on C and supported in a neighborhood U_C of C. In the case of a Lefschetz fibration, if C is a vanishing cycle then τ_C is the monodromy around the corresponding critical value. The assumption $\dim M \geq 4$ in the following is equivalent to its Lagrangian submanifolds, in particular the sphere C, being of codimension (equal to dimension) at least 2.

Theorem 1.1. (Seidel exact triangle, [42]) *Let M be a compact monotone or exact symplectic manifold with convex boundary and dimension at least four, and let C, L^0, L^1 be admissible Lagrangian branes such that C is equipped with a diffeomorphism to a sphere. Then there exists a long exact sequence*

$$HF(L^0, L^1) \longrightarrow HF(L^0, \tau_C L^1)$$
$$HF(L^0, C) \otimes HF(C, L^1).$$

Seidel's result is often referred to as a categorification of Picard-Lefschetz since by taking the Euler characteristics one essentially recovers the Picard-Lefschetz formula as in Arnold [4, Chapter I]. The Fukaya-categorical version (conjectured by Kontsevich) is developed in Seidel's book [43].

Many interesting fibrations that arise in representation theory or gauge theory (such as nilpotent slices or moduli spaces of bundles over a family of curves) have not just Morse singularities but rather Morse-Bott singularities. Here the words *Morse* and *Morse-Bott* are used in the sense of non-degeneracy of the Hessian, which in this setting is a *complex* matrix. Extensions of the Picard-Lefschetz formula to fibrations $\pi : E \to S$ with more general singularities are considered by Clemens [10], Landman [22], and many subsequent authors; the Morse-Bott situation is a particularly easy

case. Let $s_0 \in S$ be a critical value, and

$$B := \mathrm{crit}(\pi) \cap E_{s_0}$$

the critical locus in its fiber, which need no longer consist of isolated points. Let $s \in S$ be a generic nearby point. The analog of the vanishing cycle in this case is a manifold C fibering over B,

$$E_s \supset C \to B$$

consisting of points that converge to B under parallel transport, called by Clemens [10] the *vanishing bundle*. In the symplectic setting, C is a coisotropic submanifold and the map to B is a smooth submersion with maximally isotropic fibers. The monodromy τ_C, which arises from parallel transport around s_0, is a *fibered Dehn twist* as introduced in Section 2.2. Roughly speaking, a fibered Dehn twist is a Dehn twist in each fiber of a fibered neighbourhood $U \to B$ of $C \subset M$. Let $M = E_s$ and c the codimension of C. A special case of Clemens [10, Theorem 4.4] gives that the monodromy τ_C acts on a homology class $\alpha \in H(M)$ by the formula

(1) $$(\tau_C)_* \alpha = \alpha + (-1)^{(c+1)(c+2)/2} [C] \cdot [C^t] \cdot \alpha,$$

where the action of

$$[C] \cdot : H(B) \to H(M), \quad [C^t] \cdot : H(M) \to H(B)$$

is given by slant products, that is, the images of $[C]$ under

$$H(M \times B) \to \mathrm{Hom}(H(M), H(B)) \ \text{ resp. } \ \mathrm{Hom}(H(B), H(M)).$$

The main result of this paper is a categorification of the fibered Picard-Lefschetz formula (1) to the setting of Floer-Fukaya theory. In other words, we generalize Seidel's triangle to the fibered case. As before, need to assume will use suitable monotonicity conditions to ensure well-defined Floer cohomology and relative invariants arising from pseudoholomorphic quilts in [46]. However, we expect our proofs to directly generalize to a version of the exact triangle in any setting in which algebraic and analytic refinements provide well-defined Floer cohomologies.

Definition 1.2. (Spherically fibered coisotropics) A *spherically fibered coisotropic submanifold* of a symplectic manifold M is a coisotropic submanifold $C \subset M$ of codimension $c \geq 1$ such that

(a) (Fibrating) the null-foliation of C is fibrating over a symplectic base B with fiber S^c a sphere of dimension c and
(b) (Orthogonal structure group) the structure group of $p : C \to B$ is equipped with a reduction to $SO(c+1)$, that is, a principal $SO(c+1)$-bundle $P \to B$ and a bundle isomorphism $P \times_{SO(c+1)} S^c \cong C$.

Any spherically fibered coisotropic gives rise to a fibered Dehn twist $\tau_C \in \mathrm{Diff}(M, \omega)$, see Section 2. We identify C with its Lagrangian image in $B^- \times M$, where B^- denotes B with symplectic structure reversed. Let C^t denote the transpose of C in $M^- \times B$. Thus C defines *Lagrangian correspondences* from B to M and vice-versa. These correspondences fit into the framework of *quilted Floer theory* developed in [51], [52], [46], [47]. Assuming monotonicity as in in [46] (i.e. an energy-index relation for

pseudoholomorphic quilts), the correspondence C defines functors from the Fukaya category of B to that of M and vice versa. On the level of homology, the latter functor gives rise to a homomorphism between quilted Floer cohomology groups

$$(2) \qquad HF(\ldots, L^0, C^t, C, L^1, \ldots)[\dim(B)] \to HF(\ldots, L^0, L^1, \ldots),$$

where the Lagrangian correspondences

$$L^0 \subset N_0^- \times M, \quad L^1 \subset M^- \times N_1$$

are parts of generalized Lagrangian correspondences

$$\underline{L}^0 = (\ldots, L^0), \quad \underline{L}^1 = (\ldots, (L^1)^t)$$

from a point to M in the sense of [47]. In the special case of simple Lagrangian submanifolds $\underline{L}^i = L^i \subset M$, the homomorphism (2) is equivalent with $\mathbb{Z}_2$ coefficients to a homomorphism

$$HF(L^0 \times C, C^t \times L^1)[\dim(B)] \to HF(L^0, L^1).$$

The map (2) is more precisely defined in [46] and provides the "quilted chaps" map which we use to generalize the "chaps" map in Seidel's proof of the exact triangle. In the above special case, we obtain an exact triangle

$$HF(L^0, L^1) \xrightarrow{\hspace{2cm}} HF(L^0, \tau_C L^1)$$
$$HF(L^0 \times C, C^t \times L^1)[\dim(B)].$$

The precise statement including monotonicity conditions, and allowing for generalized Lagrangian correspondences, is the following, which we prove in Section 5.2. Here $\operatorname{graph}(\tau_C) \subset M^- \times M$ denotes the graph of the fibered Dehn twist τ_C.

Theorem 1.3. (Exact triangle for fibered Dehn twists) *Let M be a compact monotone or exact symplectic manifold with convex boundary, let $\underline{L}^0 = (\ldots, L^0)$ and $\underline{L}^1 = (\ldots, (L^1)^t)$ be admissible generalized Lagrangian branes in M, and let $C \subset M$ be a spherically fibered coisotropic submanifold of codimension $c \geq 2$ with base B, that is equipped with an admissible brane structure as a Lagrangian submanifold of $M^- \times B$. Then there exists an exact triangle*

$$HF(\ldots, L^0, L^1, \ldots) \xrightarrow{\hspace{2cm}} HF(\ldots, L^0, \operatorname{graph}(\tau_C), L^1, \ldots)$$
$$HF(\ldots, L^0, C^t, C, L^1, \ldots)[\dim(B)].$$

A similar triangle was developed by T. Perutz, as part of the program described in [34]. A different approach to exact triangles via Lagrangian cobordism is given in the recent work of Mak-Wu [26] who also treated the codimension one case with $\mathbb{Z}_2$ coefficients for the first time. We also treat the codimension one case below (see

Remark 5.10) although the monotonicity assumptions required in this case are more complicated.

As in Seidel's work, there is a connection with the mapping cone construction in the derived Fukaya category, which we establish in Section 7.1 as follows.

Theorem 1.4. (Derived Fukaya-categorical version of the exact triangle for fibered Dehn twists) *Let $C \subset M$ be a spherically fibered coisotropic submanifold with admissible brane structure as above. Then for every admissible generalized Lagrangian brane $\underline{L}$ in M there exists an exact triangle in the derived Fukaya category*

$$\underline{L} \longrightarrow \underline{L} \# \operatorname{graph}(\tau_C)$$

$$\underline{L} \# C^t \# C[\dim(B)].$$

Here the notation $\underline{L} \# \operatorname{graph}(\tau_C)$ and $\underline{L} \# C^t \# C$ indicates the generalized Lagrangian submanifolds in the sense of [47] formed by concatenation $\#$. For $\underline{L} = (L_0, \ldots, L_k)$ with $L_i \subset M_{i-1}^- \times M_i$ and $M_k = M$ the object $\underline{L} \# \operatorname{graph}(\tau_C)$ is equivalent to $\left(L_0, \ldots, (\operatorname{id}_{M_{k-1}} \times \tau_C)(L_k)\right)$ for suitable choices of relative spin structures. In particular $L \# \operatorname{graph}(\tau_C) \sim \tau_C(L)$ for $k = 0$. One can also write the bottom object in the exact triangle

$$L \# C^t \# C[\dim(B)] = \Phi(C)\Phi(C^t)L$$

where $\Phi(C^t), \Phi(C)$ are the A_∞ functors associated to Lagrangian correspondences constructed in [28]. The formulation of the third term as a push-pull functor makes clear that the exact triangle is the mirror partner of Horja's exact triangle in [17]. We remark that Perutz [36] proves a related exact triangle describing a symplectic version of the Gysin sequence; roughly speaking Perutz' result describes the composition of the functors for C^t, C in the opposite order as a mapping cone for the map given by multiplication of the Euler class.

We briefly outline the contents of the paper. Section 2 contains background results on fibered Dehn twists and Lefschetz-Bott fibrations. Section 3 describes various situations in which surface Dehn twists induce generalized Dehn twists on moduli spaces of flat bundles; these are mostly minor improvements of results of Seidel and Callahan. Sections 4 and 5 contain the proof of the exact triangle. Section 6 applies the triangle to moduli spaces of flat bundles to obtain generalizations of Floer's exact triangle for surgery along a knot, as well as surgery exact triangles for crossing changes in knots which have the same form as the surgery exact triangles as Khovanov [18] and Khovanov-Rozansky [20]. Finally Section 7 describes generalizations to the A_∞ setting. These are limited to the case of minimal Maslov number greater than two.

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The present paper is an updated and more detailed version of a paper the authors have circulated since 2007. The authors have unreconciled differences over the exposition in the paper, and explain their points of view at [45] and [53]. The publication in the current form is the result of a mediation.

2. Lefschetz-Bott fibrations and fibered Dehn twists

This section covers the generalization of the theory of symplectic Lefschetz fibrations to the Lefschetz-Bott case, that is, to the case that the singularities of the fibration are not isolated but still non-degenerate in the normal directions. Most of this material is covered in an unpublished manuscript of Seidel [40] and in the works of Perutz [34], [35]. For more recent appearance of fibered Dehn twists, see Chiang et al [9].

2.1. **Symplectic Lefschetz-Bott fibrations.** Lefschetz-Bott fibrations have a natural definition in the setting of holomorphic geometry: one requires the projection to be proper and Morse-Bott. In the setting of symplectic geometry, there are several analogous definitions which we discuss below. We begin with the holomorphic setting: Let S be a complex curve. A *Lefschetz fibration* over S is a complex manifold E equipped with a proper holomorphic map $\pi : E \to S$ such that π only has critical points of Morse type. That is, in local coordinates $z_1, \ldots, z_n$ on E, the map π is given by

$$\pi(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i^2.$$

A *Lefschetz-Bott fibration* over S is a complex manifold equipped with a proper holomorphic map $\pi : E \to S$ that has only Morse-Bott singularities. That is, the critical set

$$E^{\mathrm{crit}} = \{e \in E \,|\, D_e\pi = 0\}$$

is a smooth (necessarily holomorphic) submanifold and the Hessian of π is non-degenerate along the normal bundle of E^{crit}. By the parametric Morse lemma [3, p.12] for any critical point $e \in E^{\mathrm{crit}}$ there exist a neighborhood U of e and coordinates $(z_1, \ldots, z_n) : U \to \mathbb{C}^n$ such that

$$(3) \qquad \pi(z_1, \ldots, z_n) = \sum_{i=1}^{c+1} z_i^2,$$

where n is the dimension of E and $c + 1$ is the codimension of E^{crit} at e.

In our examples we will not have global complex structures on E and S (at least no canonical ones). Instead, we work with symplectic versions of Lefschetz-Bott fibrations. The definition of the symplectic version uses the following condition introduced in Perutz [34]. Let (M, ω) be a symplectic manifold equipped with an almost complex structure J and $M' \subset M$ an almost complex submanifold. The submanifold M' is said to be *normally Kähler* if a tubular neighborhood N of M' in M is foliated by J-complex normal slices $\{N_e \subset N\}, e \in M'$, such that $J|N_e$ is integrable and $\omega|N_e$ is Kähler for each e.

Definition 2.1. (a) (Symplectic fibrations) A *symplectic fibration* is a manifold E equipped with a closed two-form $\omega_E \in \Omega^2(E)$ and a fibration $\pi : E \to S$ over a smooth surface S, such that the restriction of ω_E to any fiber of π is symplectic:

$$(\omega_E(e)|_{D_e\pi^{-1}(0)})^{\dim(E)-1} \neq 0, \forall e \in E.$$

(b) (Symplectic Lefschetz-Bott fibrations) A *symplectic Lefschetz-Bott fibration* consists of

(i) a smooth manifold E equipped with a closed two-form ω_E;

(ii) a smooth, oriented surface S;

(iii) a smooth proper map $\pi : E \to S$ with critical set and values

$$E^{\mathrm{crit}} := \{e \in E \mid \mathrm{rank}(D_e\pi) < 2\}, \quad S^{\mathrm{crit}} = \pi(E^{\mathrm{crit}}) \subset S;$$

(iv) a positively oriented complex structure $j_0 \in \mathrm{End}(TS|_{\mathcal{U}})$ defined in a neighborhood $\mathcal{U} \subset S$ of the critical values S^{crit}; and

(v) an almost complex structure $J_0 \in \mathrm{End}(TE|_{\mathcal{V}})$ defined in a neighborhood $\mathcal{V} \subset E$ of the critical set E^{crit}

satisfying the following conditions:

(i) $E^{\mathrm{crit}} \subset E$ is a smooth submanifold with finitely many components;

(ii) E^{crit} is normally Kähler;

(iii) $\pi|_{\mathcal{V}} : \mathcal{V} \to \mathcal{U}$ is (J_0, j_0) holomorphic;

(iv) the normal Hessian $D^2\pi_e|T^{\otimes 2}N_e$ at any critical point is non-degenerate;

(v) ω_E is non-degenerate on $\ker(D\pi) \subset TE$;

(vi) $\omega_E|_{\mathcal{V}}$ is non-degenerate and compatible with J_0.

Remark 2.2. One natural reason to consider symplectic Lefschetz-Bott fibrations rather than symplectic Lefschetz fibrations is that the category of Lefschetz-Bott fibrations is somewhat better behaved than Lefschetz fibrations with respect to products. Suppose that $\pi_k : E_k \to S, k = 1, 2$ are (symplectic) Lefschetz-Bott fibrations such that $\mathrm{critval}(\pi_1) \cap \mathrm{critval}(\pi_2)$ is empty, where critval denotes the set of critical values. Then

$$\pi_1 \times_S \pi_2 : E_1 \times_S E_2 \to S$$

is a (symplectic) Lefschetz-Bott fibration with

$$\mathrm{critval}(\pi_1 \times_S \pi_2) = \mathrm{critval}(\pi_1) \cup \mathrm{critval}(\pi_2).$$

In particular, the fiber product of a fibration with a Lefschetz fibration is a Lefschetz-Bott fibration.

Associated to any Lefschetz-Bott fibration there is a natural parallel transport between the fibers. The usual notion of parallel transport in Lefschetz fibrations extends to the symplectic Lefschetz-Bott case. First, suppose that $\pi : E \to S, \omega_E \in \Omega^2(E)$ is a symplectic fibration with connected base S, for simplicity. The canonical symplectic connection on E is the connection defined by

(4) $$T_e^h E := \left(\ker(D_e\pi)\right)^{\omega_E} \subset T_e E.$$

Here the superscript denotes the symplectic complement with respect to ω_E. The symplectic complement has dimension 2 due to the nondegeneracy of $\omega_E|_{\ker(D_e\pi)}$.

Moreover, for any horizontal vector field $v \in \Gamma(T^h E)$ and fiber E_s we have

$$\mathcal{L}_v \omega_E|_{E_s} = (\mathrm{d}\iota_v \omega_E + \iota_v \mathrm{d}\omega_E)|_{E_s} = \mathrm{d}(\iota_v \omega_E|_{E_s}) = 0.$$

Hence, given any smooth path $\gamma : [0,1] \to S - S^{\mathrm{crit}}$ the parallel transport

$$\rho_{t,\tau} : E_{\gamma(t)} \to E_{\gamma(\tau)}$$

for any $t, \tau \in [0,1]$ is a symplectomorphism. (Also see [16, Section 1.2].) This parallel transport gives a reduction of structure group of the fibration to the symplectomorphism group of any fiber.

The notion of parallel transport above extends to parallel transport to critical fibers. Suppose that $\pi : E \to S$ is a Lefschetz-Bott fibration. The smooth locus $E - \pi^{-1}(S^{\mathrm{crit}})$ is a fibration over $S - S^{\mathrm{crit}}$ with vertical tangent spaces $T_e^v E = \ker(D_e \pi)$ and a canonical symplectic connection $T^h E \subset T(E - E^{\mathrm{crit}})$ defined by (4). Given an embedded path $\gamma : [0,1] \to E$ ending on the critical locus $\gamma(1) \in S^{\mathrm{crit}}$ parallel transport extends to a continuous map

$$(5) \qquad \rho_{t,1} : E_{\gamma(t)} \to E_{\gamma(1)}, \quad x \mapsto \lim_{\tau \to 1} \rho_{t,\tau}(x).$$

Indeed, choose a tubular neighborhood of γ. After rescaling, parallel transport becomes the gradient flow of the function $f \circ \pi$ for some coordinate function $f : S \to \mathbb{R}$ with respect to the metric $\omega_E(\cdot, J_0 \cdot)$, see [42, Lemma 1.13]. Since the critical points of π are Lefschetz-Bott, the gradient flow is hyperbolic and the limit is well-defined [44].

Vanishing thimbles and cycles for Lefschetz-Bott fibrations are defined as follows. Let $\gamma : [0,1] \to S$ be a smooth embedded path with $\gamma(1) \in S^{\mathrm{crit}}$ such that $\gamma([0,1)) \subset S - S^{\mathrm{crit}}$. Fix a connected component $B \subset E^{\mathrm{crit}} \cap E_{\gamma(1)}$ of the critical set in the endpoint fiber. The *vanishing thimble* for the path γ and component B is

$$T_{\gamma,B} = \left\{ x \in \cup_{t \in [0,1)} E_{\gamma(t)} \,\middle|\, \rho_{t,1}(x) \in B \right\} \cup B.$$

The vanishing thimble $T_{\gamma,B} \subset E$ is a smooth submanifold with boundary since it is the stable manifold of B; see [44]. The intersections

$$(6) \qquad C_t := T_{\gamma,B} \cap E_{\gamma(t)}$$

with the smooth fibers of π for $t \in [0,1)$ are called the *vanishing cycles* for the path γ.

Proposition 2.3. *Each vanishing cycle C_t from (6) is a coisotropic submanifold of the fiber $E_{\gamma(t)}$. The map $\rho_{t,1} : C_t \to B$ is smooth and gives C_t the structure of a spherically fibered coisotropic submanifold in the sense of Definition 1.2.*

Proof. The parallel transport map $\rho_{t,1}$ of (5) is a smooth fibration with fibers c-dimensional spheres and structure group $SO(c+1)$ by the stable manifold theorem, see for example [44]. The dimension c is the dimension of the fiber as well as the codimension of $C_t \subset E_{\gamma(t)}$, by the normal form (3) of π. The parallel transport can also be written as a rescaled Hamiltonian flow of $g \circ \pi$ for some coordinate function

$g : S \to \mathbb{R}$, as in the unfibered case described in [42]. Since the Hamiltonian flow preserves the symplectic form, the symplectic form vanishes on the fibers of C_t:

$$\omega_E|_{\rho_{t,1}^{-1}(b)} = 0, \quad \forall b \in B.$$

Since π is J_0-holomorphic on B, the tangent space TB (which is the null space of the Hessian of π) is J_0-invariant. Hence nondegeneracy of ω_E in a neighborhood of B implies that the restriction of ω_E to B is symplectic. Thus the total space C_t is coisotropic and the projection $\rho_{t,1} : C_t \to B$ is the null foliation as claimed. $\qquad\square$

2.2. **Fibered Dehn twists.** The symplectic Dehn twist along a Lagrangian sphere in [42] can be generalized to spherically fibered coisotropics using the associated symplectic fiber bundle construction. This construction associates to any principal bundle and Hamiltonian action with small moment map a symplectic fiber bundle.

We first set up notation for connections on principal bundles. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Recall that a *connection one-form* on a principal G-bundle P is a one-form $\alpha \in \Omega^1(P, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g}$ satisfying the following two conditions:

(a) the evaluation $\alpha(\xi_P)$ is the constant function equal to ξ for any $\xi \in \mathfrak{g}$, where $\xi_P \in \mathrm{Vect}(P)$ is the vector field generating the action of ξ, and
(b) the pull-back $g^*\alpha$ is equal to $\mathrm{Ad}(g)^{-1}\alpha$ for any $g \in G$, where the adjoint action is on the values of α.

Let $\ker(\alpha) \subset TP$ denote the *horizontal subbundle* of vectors whose fiber at a point point $p \in P$ is the space of vectors annihilated by α_p. The tangent bundle TP admits a splitting into horizontal and vertical parts

$$TP \cong \ker(\alpha) \oplus \ker(D\pi)$$

invariant under the group action. Denote by $\mathcal{A}(P)$ the space of connection one-forms on P; it is an affine space modelled on $\Omega^1(C, P(\mathfrak{g}))$.

Given a principal bundle with connection and a Hamiltonian action, the associated fiber bundle has a natural two-form defined as follows. To set up notation, suppose that the following are given:

- a *base symplectic manifold* (B, ω_B);
- a *principal bundle* $\pi : P \to B$ with structure group G;
- a *fiber symplectic manifold* (F, ω_F) equipped with a Hamiltonian G-action with moment map $\Phi_F : F \to \mathfrak{g}^\vee := \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$; and
- a *connection one-form* $\alpha \in \Omega^1(P, \mathfrak{g})^G$.

Denote by π_1, π_2 the projections to the factors of $P \times F$. The *minimally coupled form* on $P \times F$ is

$$(7) \qquad \omega_{P \times F, \alpha} = \pi_1^* \pi^* \omega_B + \pi_2^* \omega_F + \mathrm{d}\langle \pi_1^* \alpha, \pi_2^* \Phi_F \rangle \in \Omega^2(P \times F).$$

Theorem 2.4. (Symplectic associated fiber bundles) *(see e.g. [15]) Let P be a principal G-bundle, F a Hamiltonian G-manifold and α a connection one-form as above.*

(a) *The minimally coupled form $\omega_{P \times F, \alpha}$ descends to a closed two-form*

$$\omega_{P(F), \alpha} \in \Omega^2(P(F)), \quad P(F) := P \times_G F.$$

The form $\omega_{P(F), \alpha}$ is non-degenerate on the fibers in a neighborhood

$$P(F)_\epsilon := P(\{|\Phi_F| < \epsilon\})$$

of the associated bundle to the zero level set $P(\Phi_F^{-1}(0)) \subset P(F)$. Here $|\Phi_F|$ denotes the norm of Φ_F with respect to an invariant metric on $\mathfrak{g} \cong \mathfrak{g}^\vee$. Hence $P(F)_\epsilon$ is a symplectic fiber bundle over B for sufficiently small $\epsilon > 0$.

(b) *If $\Phi_F^{-1}(0)$ is smooth, then $P(\Phi_F^{-1}(0))$ is a smooth submanifold of $P(F)_\epsilon$ with coisotropic fibers.*

(c) *Given two choices of connection $\alpha_j, j = 0, 1$, there exists an isomorphism of symplectic fiber bundles from $(P(F)_\epsilon, \omega_{P(F), \alpha_0})$ to $(P(F)_\epsilon, \omega_{P(F), \alpha_1})$ for sufficiently small $\epsilon > 0$.*

(d) *The association $(F, \omega_F, \Phi_F) \to (P(F)_\epsilon, \omega_{P(F)})$ is functorial in the sense that any isomorphism of Hamiltonian G-manifolds $(F_0, \omega_{F_0}, \Phi_{F_0})$ to $(F_1, \omega_{F_1}, \Phi_{F_1})$ induces an isomorphism of symplectic fiber bundles $P(F_0)_\epsilon \to P(F_1)_\epsilon$.*

Example 2.5. (Associated bundles with cotangent-sphere fibers) We are mainly interested in the following special case of the general construction. For any integer $c \geq 1$, let S^c denote a sphere of dimension c and $T^\vee S^c$ its cotangent bundle. Consider $T^\vee S^c$ with canonical symplectic form $\omega_{T^\vee S^c}$ and the canonical $SO(c+1)$-action. The action is Hamiltonian with a moment map $\Phi_{T^\vee S^c}$ whose zero level set is S^c. Thus for any principal $SO(c+1)$-bundle $\pi : P \to B$ the associated fiber bundle construction yields a symplectic fiber bundle $P(T^\vee S^c)_\epsilon$ over B. By functoriality, any automorphism τ of $(T^\vee S^c, \omega_{T^\vee S^c}, \Phi_{T^\vee S^c})$ induces a bundle isomorphism $\tau_{P(T^\vee S^c)} : P(T^\vee S^c) \to P(T^\vee S^c)$. The latter is an isomorphism of symplectic fiber bundles on $P(T^\vee S^c)_\epsilon$.

The notion of Dehn twist is most familiar from Riemann surface theory, where a Dehn twist denotes a diffeomorphism obtained by twisting around a circle on a handle. In [42] Seidel introduces a generalized notion of Dehn twist which is a symplectomorphism around a Lagrangian sphere, called a *generalized Dehn twist*. The symplectomorphisms we consider here are further generalized by allowing the twists to be fibered, so that the vanishing cycles are fibered coistropics. To save space, we call these simply fibered Dehn twists. We begin with the local model introduced by Seidel [42].

Definition 2.6. (Model Dehn twist, Seidel [42, Lemma 1.8]) For any sphere S^c, a *model Dehn twist* τ_{S^c} along the zero section S^c in the cotangent bundle $T^\vee S^c$ is a compactly supported symplectomorphism equal to the antipodal map on S^c, given as follows. Let $\zeta \in C^\infty(\mathbb{R})$ be a function satisfying

(8) $\zeta(t) = 0 \quad$ for $t \geq \epsilon \quad$ and $\quad \zeta(-t) = \zeta(t) - t \quad$ for all $t \in \mathbb{R}$.

In particular, ζ is compactly supported and $\zeta'(0) = 1/2$. Fix the standard Riemannian structure on S^c and let

$$T^\vee S^c \to \mathbb{R}, \quad v \mapsto |v|$$

denote the Riemannian norm. The norm is smooth on the complement of the zero section and similarly for composition with ζ. The time 2π flow of $v \mapsto \zeta(|v|)$ extends to a smooth symplectomorphism τ_{S^c} of $T^\vee S^c$. The flow acts on the zero section by the antipodal map given by

$$\tau_{S^c}|_{S^c} : S^c \to S^c, \quad v \mapsto -v.$$

Furthermore τ_{S^c} is $SO(c+1)$-equivariant and preserves the moment map for the $SO(c+1)$-action. Any two model Dehn twists given by different choices of ζ are equivalent up to symplectomorphism generated by a compactly supported Hamiltonian.

We construct fibered Dehn twists along spherically fibered coisotropics as follows.

Definition 2.7. Let (M, ω) be a symplectic manifold and $C \subset M$ a spherically fibered coisotropic submanifold of codimension $c \geq 1$, fibering $C \to B$ over a symplectic manifold (B, ω_B), as in Definition 1.2.

(a) (Coisotropic embedding) Recall that C is diffeomorphic to an associated fiber bundle $P(S^c) := P \times_{SO(c+1)} S^c$, for some principal $SO(c+1)$-bundle $\pi : P \to B$. By the coisotropic embedding theorem [15, p.315], a neighborhood of C in M is symplectomorphic to a neighborhood of the zero section in $P(T^\vee S^c)_\epsilon$ as in Theorem 2.4.

(b) (Model fibered Dehn twists) Any $SO(c+1)$-equivariant model Dehn twist $\tau_{T^\vee S^c} : T^\vee S^c \to T^\vee S^c$ induces a symplectomorphism

$$\tau_{P(T^\vee S^c)} : P(T^\vee S^c)_\epsilon \to P(T^\vee S^c)_\epsilon$$

by functoriality of the associated symplectic fiber bundle construction as in Example 2.5. Given a symplectomorphism ϕ of a neighborhood U of $C \subset M$ with $P(T^\vee S^c)_\epsilon$ we define a symplectomorphism $\tau_C : M \to M$ by $\tau_{P(T^\vee S^c)}$ on the neighborhood of C and the identity outside:

$$\tau_C|_U = \phi^{-1} \tau_{T^\vee P(S^c)}|_{\phi(U)} \phi, \quad \tau_C|_{M-U} = \mathrm{Id}_M.$$

We call τ_C a *model fibered Dehn twist along C*.

(c) (Fibered Dehn twists) A symplectomorphism τ_C of M is called a *fibered Dehn twist along C* if τ_C is Hamiltonian isotopic to a model Dehn twist.

Remark 2.8. The Hamiltonian isotopy class of a fibered Dehn twist is independent of the choice of local model and fibered Dehn twist used in its construction: Any two local models for a fibered coisotropic are isotopic, by a family version of Moser's construction. This fact implies that any two fibered Dehn twists $\tau_{C,0}, \tau_{C,1}$ defined using different local models and model twists may be connected by a family $\tau_{C,t}$. The vector field $v_t := (\tau_{C,t}^{-1})_* \frac{d}{dt} \tau_{C,t} \in \mathrm{Vect}(M)$ vanishes on C and is necessarily Hamiltonian in a neighborhood of C.

2.3. **Equivariant fibered Dehn twists.** In this section we discuss the interaction of equivariant fibered Dehn twists with symplectic reduction. Let G be a compact connected Lie group and (M, ω) a symplectic G-manifold. A *spherically fibered coisotropic G-submanifold* is an invariant coisotropic submanifold $C \subset M$ of codimension $c \geq 1$ such that there exists

- a principal $SO(c+1)$-bundle $\pi : P \to B$ equipped with an action of G by bundle automorphisms (i.e. $SO(c+1)$-equivariant diffeomorphisms) and
- a G-equivariant bundle isomorphism $P \times_{SO(c+1)} S^c \cong C$, where the G-action is induced by the action on the first factor.

Given such a coisotropic C, one obtains a G-equivariant Dehn twist on $P(T^\vee S^c)$ by the G-equivariant version of the associated symplectic fiber bundle construction. One obtains a *G-equivariant model fibered Dehn twist* on M via the G-equivariant coisotropic embedding theorem. A G-equivariant symplectomorphism $\tau \in \mathrm{Diff}(M,\omega)$ is a *G-equivariant fibered Dehn twist* if τ is equivariantly Hamiltonian isotopic to a model Dehn twist, that is, Hamiltonian isotopic via a symplectomorphism generated by a family of *G-invariant* Hamiltonians.

We now show that equivariant fibered Dehn twists give rise to fibered Dehn twists in symplectic quotients. Let (M,ω) be a Hamiltonian G-manifold with moment map $\Phi : M \to \mathfrak{g}^\vee$. The symplectic quotient of M by G is

$$M /\!\!/ G := \Phi^{-1}(0)/G.$$

Assuming that G acts freely on $\Phi^{-1}(0)$, $M/\!\!/ G$ is a symplectic manifold with unique symplectic form that lifts to the restriction of the symplectic form to $\Phi^{-1}(0)$.

Lemma 2.9. *Let (M,ω) be a Hamiltonian G-manifold with moment map Φ. Let $C \subset M$ be a G-invariant coisotropic submanifold. Suppose that 0 is a regular value of Φ, the action of G on $\Phi^{-1}(0)$ is free, and C intersects $\Phi^{-1}(0)$ transversally. The quotient*

$$C /\!\!/ G := (C \cap \Phi^{-1}(0))/G$$

is a coisotropic submanifold of $M /\!\!/ G$.

Proof. The transversality $TC \pitchfork \ker D\Phi$ implies $(\ker D\Phi)^\omega \cap TC^\omega = \{0\}$. It follows that

$$(\ker D\Phi \cap TC)^\omega = (\ker D\Phi)^\omega + TC^\omega = (\ker D\Phi)^\omega \oplus TC^\omega.$$

Now $(\ker D\Phi)^\omega \cong \mathfrak{g}$ is the tangent space to the G-orbits and contained in TC. On the other hand, $TC^\omega \subseteq TC$ since C is coisotropic. Hence

$$(\ker D\Phi \cap TC)^\omega \cap (\ker D\Phi) \subseteq \ker D\Phi \cap TC.$$

This implies $T(C/\!\!/G)^\omega \subseteq T(C/\!\!/G)$. $\qquad\qquad\square$

Lemma 2.10. *Suppose that $C \subset M$ is a spherically fibered G-coisotropic over a base B where M is a Hamiltonian G-manifold with moment map Φ. Then Φ is constant on the fibers of C and the induced action of G on B is Hamiltonian with moment map $\Phi_B : B \to \mathfrak{g}^\vee$ the unique map satisfying $p^*\Phi_B = \Phi|C$.*

Proof. By assumption, the action of G on P is $SO(c+1)$-equivariant and so induces an action on B. For any $\xi \in \mathfrak{g}$ the infinitesimal action $\xi_M \in \mathrm{Vect}(M)$ is tangent to C. Hence $L_v \langle \Phi, \xi \rangle = \omega(\xi_M, v) = 0$ for all fiber tangent vectors $v \in T^{\mathrm{vert}}C = TC^\omega$. It follows that Φ is constant on the fibers of $p : C \to B$. So Φ induces a map $\Phi_B : B \to \mathfrak{g}^\vee$, satisfying $\mathrm{d}\langle \Phi_B, \xi \rangle = \iota(\xi_B)\omega_B$ for all Lie algebra vectors $\xi \in \mathfrak{g}$ as claimed. $\qquad\qquad\square$

Remark 2.11. (Quotients of spherically fibered coisotropics are spherically fibered) Suppose that in the setting of Lemma 2.10, G also acts freely on $\Phi_B^{-1}(0)$. It follows that $B/\!\!/G = \Phi_B^{-1}(0)/G$ is a smooth symplectic quotient. Then the null foliation on $C/\!\!/G$ fibers over $B/\!\!/G$: If $p : C \to B$ is the projection then

$$C/\!\!/G := (C \cap \Phi^{-1}(0))/G = p^{-1}(\Phi_B{}^{-1}(0))/G \xrightarrow{\ p\ } \Phi_B{}^{-1}(0)/G =: B/\!\!/G.$$

Define $\tilde{P} = (P|\Phi_B{}^{-1}(0))/G$; this quotient is a principal $SO(c+1)$-bundle over $\tilde{B} = B/\!\!/G$. We have a bundle isomorphism $\tilde{P} \times_{SO(c+1)} S^c \cong C/\!\!/G$. It follows that $C/\!\!/G$ is spherically fibered in the sense of Definition 1.2.

In this setting every G-equivariant fibered Dehn twist along C descends to a fibered Dehn twist of $M/\!\!/G$ along $C/\!\!/G$:

Theorem 2.12. *Let (M, ω) be a Hamiltonian G-manifold with moment map Φ such that 0 is a regular value of Φ and the action of G on $\Phi^{-1}(0)$ is free. Let $C \subset M$ be a spherically fibered G-coisotropic over a base B. Suppose that C intersects $\Phi^{-1}(0)$ transversally, and that the induced action of G on the base $\Phi_B^{-1}(0) \subset B$ is free. Let $\tau_C \in \mathrm{Diff}(M, \omega)$ be a G-equivariant fibered Dehn twist along C. Then the symplectomorphism*

$$[\tau_C] : M/\!\!/G \to M/\!\!/G, \quad [m] \mapsto [\tau_C(m)]$$

is a fibered Dehn twist $[\tau_C] =: \tau_{C/\!\!/G}$ along $C/\!\!/G$.

Proof. By definition τ_C is Hamiltonian isotopic to a model Dehn twist on $P(T^\vee S^c)$ given by $\tau_C^0 : [p, v] \mapsto [p, \tau_{S^c}(v)]$. The latter is G-equivariant since the G-action commutes with the $SO(c+1)$ action. Any G-equivariant local model $P(T^\vee S^c) \to M$ induces a local model given by a symplectomorphism of a neighborhood of the zero section in $\tilde{P}(T^\vee S^c)$ to $M/\!\!/G$. One obtains from the local model a Dehn twist

$$\tau_{C/\!\!/G}^0 : [\tilde{p}, v] \mapsto [\tilde{p}, \tau_{S^c}(v)]$$

on $\tilde{P}(T^\vee S^c)$ along $C/\!\!/G$. The equivariant Hamiltonian isotopy of τ_C to τ_C^0 induces a Hamiltonian isotopy of $\tau_{C/\!\!/G}$ to $\tau_{C/\!\!/G}^0$. This completes the proof. $\qquad\square$

2.4. **Lefschetz-Bott fibrations associated to fibered Dehn twists.** In this section we explain that any fibered Dehn twist appears as the monodromy of a symplectic Lefschetz-Bott fibration. Conversely, the monodromy of a symplectic Lefschetz-Bott fibration is given by a fibered Dehn twist up to isotopy by Theorem 2.14 of Perutz [34] recalled below. (Theorem 2.14 is not used in this paper; we mention it only for its conceptual importance linking Lefschetz-Bott fibrations and fibered Dehn twists.)

Proposition 2.13. *Let M be a symplectic manifold, $C \subset M$ a spherically fibered coisotropic, and $\tau_C : M \to M$ a fibered Dehn twist around C. There is a* standard *Lefschetz-Bott fibration E_C with generic fiber M and symplectic monodromy τ_C.*

Proof. Let $p : C \to B$ denote the fibration, and $P \to B$ the associated $SO(c+1)$-bundle. Recall from [42] the symplectic Lefschetz fibration associated to a model

Dehn twist. Given the standard representation of $SO(c+1)$ on $V = \mathbb{C}^{c+1}$ we have a vector bundle

$$P(V) := (P \times V)/SO(c+1) \to B.$$

Let

$$\omega_V \in \Omega^2(V), \quad \Phi_V : V \to \mathfrak{so}(c+1)^\vee$$

denote the symplectic form and moment map for the $SO(c+1)$-action induced from the identification $V \to T^\vee \mathbb{R}^{c+1}$. The associated symplectic fiber bundle construction above produces a closed form $\omega_{P(V)}$ on $P(V)$, equal to ω_B on B and equal to ω_V on the fiber V. The map

$$(9) \qquad\qquad \pi_V : \mathbb{C}^{c+1} \to \mathbb{C}, \quad (z_0, \dots, z_c) \mapsto \sum_{i=0}^c z_i^2$$

is $SO(c+1)$-invariant and has a single Morse singularity. By the associated bundle construction π_V induces the structure of a Lefschetz-Bott fibration on a neighborhood of the zero section of $P(V)$ over $\mathbb{C}$. Let $S^c \subset \mathbb{R}^{c+1} \subset V$ denote the unit sphere and

$$(10) \qquad\qquad T_z := \sqrt{z} S^c \subset V, \quad T := \bigcup_{z \in \mathbb{C}} T_z.$$

By [42, (1.17)] the symplectic form on V can be changed slightly so that the symplectic monodromy around 0 is a Dehn twist along S^c. By [42, 1.13] there exists an isomorphism of Hamiltonian $SO(c+1)$-manifolds

$$V - T \to \mathbb{C} \times (T^\vee S^c - S^c).$$

By the coisotropic embedding theorem, a neighborhood of C in M is symplectomorphic to the fiber bundle $P(U)$, where U is a neighborhood of the zero section in $T^\vee S^c$. It follows that $P(V) - P(T)$ is symplectomorphic to $P(\mathbb{C} \times T^\vee S^c - S^c)$ in a neighborhood of $P(T)$ resp. $P(\mathbb{C} \times S^c)$. By replacing a neighborhood of $\mathbb{C} \times C$ in $\mathbb{C} \times M$ with a neighborhood of $P(T)$ in $P(V)$, one obtains a Lefschetz-Bott fibration $E_C \to \mathbb{C}$ with monodromy τ_C. $\qquad\square$

Theorem 2.14. [34, Theorem 2.19] *Suppose that $\pi : E \to S$ is a symplectic Lefschetz-Bott fibration, and $s_0 \in S^{\mathrm{crit}}$ is such that $\pi^{-1}(s_0) \cap E^{\mathrm{crit}}$ has a unique connected component. Then the symplectic monodromy around s_0 is a fibered Dehn twist.*

2.5. **Further examples of fibered Dehn twists.** Fibered Dehn twists are often induced by flows of components of moment maps. First let $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ denote the circle group. We identify the Lie algebra of $U(1)$ with $\mathbb{R}$ via division by $2\pi i$. The integers $\mathbb{Z} = \exp^{-1}(1)$ are then the coweight lattice. Let (M, ω) be a symplectic manifold, and $M_0 \subset M$ an open subset equipped with a free Hamiltonian action of $U(1)$ with moment map $\Phi : M_0 \to (c_-, c_+)$.

Proposition 2.15. (Fibered Dehn twists via Hamiltonian circle actions) *Let $\psi \in C^\infty[c_-, c_+]$ be a function such that $\psi' = 1$ on a neighborhood of c_+ and $\psi' = 0$ on a neighborhood of c_-. Then the time one flow of $\psi \circ \Phi$ on M_0 extends to a smooth flow on M equal to the identity on the complement of M_0 and the extension is a*

fibered Dehn twist along $\Phi^{-1}(c)$ *for any* $c \in (c_-, c_+)$. *If* M_0 *is a Hamiltonian G-manifold for a compact Lie group G and* ψ *is G-invariant then this fibered Dehn twist is equivariant.*

Proof. Let ψ be as in the statement of the Proposition. We have

$$\mathrm{d}(\psi \circ \Phi) = \psi' \mathrm{d}\Phi = \iota \left(\psi' \frac{\partial}{\partial \theta} \right) \omega \in \Omega^1(M_0),$$

where $\frac{\partial}{\partial \theta} \in \mathrm{Vect}(M_0)$ is the generating vector field for the action. Hence the flow of $\psi \circ \Phi$ is rotation by the angle $2\pi\psi'$. The rest is immediate from the definitions. $\square$

Remark 2.16. (Standard Dehn twists as symplectic Dehn twists) Standard Dehn twists of complex curves arise from the construction in 2.15 as follows. Suppose that M is a complex curve and $C \to M$ an embedded circle equipped with an identification $C \cong S^1$. Choose an area form ω_M on M. Since $C \subset M$ is Lagrangian, by the Lagrangian embedding theorem there exists a tubular neighborhood $M_0 = C \times (c_-, c_+)$ on which the symplectic form is standard. Then the $U(1)$ action by rotation on the left factor of M_0 is free, and the projection Φ on the second factor is a moment map. For any ψ with the properties in Proposition 2.15, the flow of $\psi \circ \Phi$ is a standard Dehn twist.

Next we consider Dehn twists induced by flows of moment maps of $SU(2)$-actions. We fix a metric on the Lie algebra $\mathfrak{su}(2)$ so that non-zero elements ξ with $\exp(\xi) = 1$ have minimal length 1.

Proposition 2.17. (Fibered Dehn twists via Hamiltonian $SU(2)$-actions) *Suppose that* (M, ω, Φ) *is a Hamiltonian* $SU(2)$-*manifold with moment map* $\Phi : M \to \mathfrak{su}(2)^\vee \cong \mathfrak{su}(2)$ *and the stabilizer* H *of the action of* $SU(2)$ *on any point in* $\Phi^{-1}(0)$ *is trivial resp. isomorphic to* $U(1)$. *Let* $\psi \in C_c^\infty[0, \infty)$ *be a compactly-supported function such that* $\psi' = 1/2$ *in a neighborhood of* 0. *Then* $\Phi^{-1}(0)$ *is a spherically fibered coisotropic of codimension 3 resp. 2 and the flow of* $\psi \circ |\Phi|$ *is a Dehn twist along* $\Phi^{-1}(0)$.

Proof. The zero level set $P := \Phi^{-1}(0)$ is a G-equivariant G/H-bundle over the symplectic quotient $M /\!\!/ G$, by the assumption on stabilizers and existence of local slices. We identify $G/H \cong S^c$ with $c = 3$ resp. $c = 2$, in the trivial stabilizer resp. $U(1)$-stabilizer case. We show that P is induced from an $SO(c+1)$ principal bundle and that the flow of $\psi \circ |\Phi|$ is obtained by a Dehn twist by the symplectic fiber bundle construction.

Consider the case of trivial stabilizers. By the equivariant coisotropic embedding theorem, there exists an equivariant symplectomorphism of a neighborhood of $\Phi^{-1}(0)$ in M with a neighborhood of the zero section in

$$(T^\vee G) \times_G P \cong P \times \mathfrak{g}^\vee$$

where the quotient is by the diagonal action $g(v, p) = (R_{g^{-1}, *}v, gp)$ and $R_{g^{-1}, *} : T^\vee G \to T^\vee G$ is induced by the right action of G. The moment map for the action in the local model is

$$\Phi : (T^\vee G) \times_G P \to \mathfrak{g}^\vee, \quad [v, p] \mapsto \Phi_{T^\vee G}(v)$$

where Φ is the moment map for the left action of G on $T^\vee G$. The norm

$$|\Phi| : (T^\vee G) \times_G P \to \mathbb{R}_{\geq 0}, \quad ([v,p]) \mapsto |v|$$

is identified with the norm on the fibers $T^\vee S^c$. Thus the function $\psi \circ |\Phi|$ is that in the definition of fibered Dehn twist. The claim follows.

In the case of circle stabilizers, denote the fixed point set of the action of the circle subgroup H of diagonal matrices by

$$P^H \subset P, \quad \{hp = p \mid \forall h \in H\}.$$

The set P^H is a double cover of $M /\!\!/ G$, since there are two H-fixed points (the poles) in each fiber $G/H \cong S^2$. The local model in this case is

$$T^\vee(G/H) \times_{N(H)} P^H \cong G \times_{N(H)} (P^H \times (\mathfrak{g}/\mathfrak{h})^\vee).$$

Since the center $Z \subset N(H)$ acts trivially on P, we may write

$$T^\vee(G/H) \times_{N(H)} P^H \cong T^\vee(G/H) \times_{G/Z} ((G/Z) \times_{N(H)} P^H)$$

where now $G/Z \times_{N(H)} P^H$ is a $G/Z \cong SO(3)$ bundle. The moment map is given by

$$\Phi : (T^\vee(G/H)) \times_G P \to \mathfrak{g}^\vee, \quad [v,p] \mapsto \Phi_{T^\vee(G/H)}(v)$$

and the norm is again the Riemannian norm of the cotangent vector in the local model. $\qquad\square$

3. Fibered Dehn twists on moduli spaces of flat bundles

This section describes a natural collection of fibered Dehn twists on moduli spaces of flat bundles, which are our motivating examples.

3.1. **Moduli spaces of flat bundles.** We first recall the construction of symplectic structures on moduli spaces of flat bundles on surfaces with markings, that is, flat bundles on the complement of the markings with holonomies around them in fixed conjugacy classes.

Definition 3.1.   (a) (Conjugacy classes in compact 1-connected Lie groups) Let G be a simple compact, 1-connected Lie group, with maximal torus T and Weyl group $W = N(T)/T$. Let $\mathfrak{g}, \mathfrak{t}$ denote the Lie algebras of G and T. We choose a *highest root* $\alpha_0 \in \mathfrak{t}^\vee$ and *positive closed Weyl chamber* $\mathfrak{t}_+ \subset \mathfrak{t}$. Conjugacy classes in G are parametrized by the Weyl alcove

$$\mathfrak{A} := \{\xi \in \mathfrak{t}_+, \ \alpha_0(\xi) \leq 1\},$$

see [37]. For any $\mu \in \mathfrak{A}$, we denote by

$$\mathcal{C}_\mu = \{g \exp(\mu) g^{-1}, \ g \in G\}$$

the corresponding conjugacy class. Inverting each conjugacy class defines an involution

(11) $$* : \mathfrak{A} \to \mathfrak{A}, \quad \mathcal{C}_{*\mu} = \mathcal{C}_\mu^{-1}.$$

In the case $G = SU(2)$, we identify $\mathfrak{t} \cong \mathbb{R}$ and $\mathfrak{A} \cong [0, 1/2]$ so that

(12) $$\mathcal{C}_\mu = \{\mathrm{Ad}(g) \, \mathrm{diag}(\exp(2\pi i \mu), -\exp(2\pi i \mu))\}.$$

In particular

$$\mathcal{C}_{1/4} = \{A \in SU(2) \mid \mathrm{Tr}(A) = 0\}$$

is the conjugacy class of *traceless* elements of $SU(2)$.

(b) (Marked surfaces) By a *marked surface* we mean a compact oriented connected surface X equipped with a collection of distinct points

$$\underline{x} = (x_1, \ldots, x_n) \in X^n$$

and a collection of *labels*

$$\underline{\mu} = (\mu_1, \ldots, \mu_n) \in \mathfrak{A}^n.$$

For simplicity, we denote such a surface $(X, \underline{\mu})$.

(c) (Holonomies) Let $P \to X$ be a G-bundle equipped with a flat connection $A \in \mathcal{A}(P)$. Parallel transport around loops in X gives rise to a *holonomy representation*

$$\pi_1(X) \to G.$$

In particular, for any point $x \in X$ a small loop around x defines a conjugacy class in $\pi_1(X)$, obtained by joining the loop to a base point, and the holonomy around x is well-defined up to conjugacy.

(d) (Moduli spaces of bundles on marked surfaces) Let $(X, \underline{\mu})$ be a marked surface. Let $M(X, \underline{\mu})$ denote the moduli space of isomorphism classes of flat G-bundles on $X - \{x_1, \ldots, x_n\}$ with holonomy around x_i conjugate to $\exp(\mu_i)$, for each $i = 1, \ldots, n$, see e.g. Meinrenken-Woodward [32]. If $M(X, \underline{\mu})$ contains no reducible bundles (bundles with non-central automorphisms) then $M(X, \underline{\mu})$ is a compact symplectic orbifold.

Remark 3.2.   (a) (Action of central bundles) Let Z denote the center of G. Let $M_Z(X)$ denote the moduli space of Z-bundles on X with trivial holonomy around the markings. The group multiplication on Z induces a group structure on $M_Z(X)$, isomorphic to Z^{2g} where g is the genus of X. The action of Z on G induces a symplectic action of $M_Z(X)$ on $M(X, \underline{\mu})$

$$M_Z(X) \times M(X, \underline{\mu}) \to M(X, \underline{\mu})$$

corresponding to twisting the holonomies by elements of Z.

(b) (Combining central markings) A label $\mu \in \mathfrak{A}$ is *central* if $\exp(\mu)$ lies in the center Z of G. In the case $G = SU(r)$, the central labels are the vertices of the alcove $\mathfrak{A}$. Several central labels may be combined into a single central label as follows: Suppose that $\lambda_1 \in \mathfrak{A}$ resp. $\lambda_2 \in \mathfrak{A}$ are labels corresponding to $z_1, z_2 \in Z$, and $\lambda_{12} \in \mathfrak{A}$ is the label that corresponds to $z_1 z_2 \in Z$. Then there is a symplectomorphism

$$M(X, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \to M(X, \lambda_{12}, \lambda_3, \ldots, \lambda_n).$$

This follows immediately from the description of the moduli space as representations of the fundamental group as in (17).

Remark 3.3. (Moduli of flat bundles as a symplectic quotient by the loop group) The moduli space of flat bundles may be realized as the symplectic quotient of the moduli space of framed bundles on a cut surface described in [32].

First we introduce notation for the cut surface. Let $Y \subset X$ be an embedded *circle*, that is, a compact, oriented, connected one-manifold, disjoint from the markings $\underline{x}$. Let X_{cut} denote the surface obtained from X by cutting along Y as in Figure 1, with boundary components $(\partial X_{\text{cut}})_j \cong S^1, j = 1, 2$. The cut surface X_{cut} may be disconnected or connected depending on whether Y is separating.

The moduli space of *framed* flat bundles on the cut surface naturally has an action of two copies of the loop group, acting by changing the framings. Let $M(X_{\text{cut}}, \underline{\mu})$ be the moduli space of flat bundles with framings (trivializations) on the boundary of X_{cut}:

$$M(X_{\text{cut}}, \underline{\mu}) = \{P \to X_{\text{cut}}, \ A \in \mathcal{A}(P), \ \phi : P|\partial X_{\text{cut}} \to \partial X_{\text{cut}} \times G \mid \text{curv}(A) = 0\}/\sim.$$

Equivalently, in the case of simply-connected structure group, $M(X_{\text{cut}}, \underline{\mu})$ is the quotient of flat connections by gauge transformations which vanish on the boundary. Working in suitable spaces of completions the moduli space $M(X_{\text{cut}}, \underline{\mu})$ is a symplectic Banach manifold, with symplectic form given by the usual pairing of one forms and integration. Let $LG = \text{Map}(S^1, G)$ denote the loop group of G, with multiplication given by pointwise multiplication. Any element $g \in LG^2$ acts on $M(X_{\text{cut}}, \underline{\mu})$ by changing the framing:

$$g(P, A, \phi) = (P, A, (1 \times g)\phi).$$

The moment map for the action of LG^2 is restriction to the boundary

$$(13) \qquad (\Phi_1, \Phi_2) : M(X_{\text{cut}}, \underline{\mu}) \to \Omega^1(S^1, \mathfrak{g})^2, \quad A \mapsto \tau_\phi(A|\partial X_{\text{cut}})$$

where $\tau_\phi : \mathcal{A}(P|\partial X_{\text{cut}}) \to \Omega^1(S^1, \mathfrak{g})^2$ is the parametrization of the space of connections induced by ϕ. Since the orientations on the two extra boundary circles are opposite, the diagonal action of LG has moment map Φ given by the difference of the moment maps for each boundary component:

$$\Phi = \Phi_1 - \Phi_2 : M(X_{\text{cut}}, \underline{\mu}) \to \Omega^1(S^1, \mathfrak{g}).$$

Taking the quotient by the diagonal loop group action recovers the moduli space for the original surface: By [32] $M(X, \underline{\mu})$ is naturally symplectomorphic (on the smooth locus) to the symplectic quotient of $M(X_{\text{cut}}, \underline{\mu})$ by the diagonal action of LG, that is,

$$M(X, \underline{\mu}) \cong M(X_{\text{cut}}, \underline{\mu}) /\!\!/ LG.$$

In particular, the symplectic structure on $M(X, \underline{\mu})$ descends from a symplectic structure on $M(X_{\text{cut}}, \underline{\mu})$.

Locally the loop group actions admit finite dimensional slices and so the infinite-dimensional quotients above are equivalent to finite-dimensional quotients [32]. In particular, let $\mathfrak{A}^\circ \subset \mathfrak{A}$ denote the interior of the alcove. For example, for $SU(2)$, we have as in (12)

$$\mathfrak{A} \cong [0, 1/2], \quad \mathfrak{A}^\circ \cong (0, 1/2).$$

Let

$$M(X,\underline{\mu})^\circ = \{[A] \in M(X,\underline{\mu}) \mid A|_U = \xi \mathrm{d}\theta, \xi \in \mathfrak{A}^\circ\}$$

be the subset consisting of connections of "standard form" $\xi \mathrm{d}\theta, \xi \in \mathfrak{A}^\circ$ in a neighborhood U of the circle Y. Similar let

$$(14)\quad M(X_{\mathrm{cut}},\underline{\mu})^\circ = \Phi^{-1}((\mathfrak{A}^\circ)^2) = \left\{[A] \in M(X_{\mathrm{cut}},\underline{\mu}) \;\middle|\; \begin{array}{l} \forall k \in \{0,1\},\ \exists \xi_k \in \mathfrak{A}^\circ \\ A|(\partial X_{\mathrm{cut}})_k = \xi_k \mathrm{d}\theta_k \end{array}\right\}$$

The locus $M(X_{\mathrm{cut}},\underline{\mu})^\circ$ is an open subsets of $M(X_{\mathrm{cut}},\underline{\mu})$, dense if non-empty, which is the quotient by the diagonal action of the maximal torus T:

$$(15)\qquad\qquad M(X,\underline{\mu})^\circ = M(X_{\mathrm{cut}},\underline{\mu})^\circ /\!/ T;$$

see [32]. More generally, for any face σ of the alcove let LG_σ denote the stabilizer of any point in σ. The stabilizer LG_σ is isomorphic to a finite-dimensional subgroup of G via evaluation at a base point. Denote by

$$\mathfrak{A}_\sigma := \bigcup_{\overline{\tau} \supset \sigma} LG_\sigma \tau$$

the slice for the action of LG on $L\mathfrak{g}^\vee$ at σ. Then

$$M(X_{\mathrm{cut}},\underline{\mu})^\sigma := (\Phi_1 \times \Phi_2)^{-1}(\mathfrak{A}_\sigma \times \mathfrak{A}_\sigma)$$

is a Hamiltonian LG_σ^2 space whose quotient

$$(16)\qquad\qquad M(X,\underline{\mu})^\sigma := M(X_{\mathrm{cut}},\underline{\mu})^\sigma /\!/ LG_\sigma$$

is a dense (if non-empty) open subset of $M(X_{\mathrm{cut}},\underline{\mu})$. As the face σ varies the collection $M(X,\underline{\mu})^\sigma$ covers $M(X_{\mathrm{cut}},\underline{\mu})$:

$$M(X_{\mathrm{cut}},\underline{\mu}) = \bigcup_{\sigma \subset \mathfrak{A}} M(X,\underline{\mu})^\sigma.$$

3.2. **Symplectomorphisms induced by Dehn twists.** Any Dehn twist on a marked surface induces a symplectomorphism of the moduli space of flat bundles. In this section we explicitly describe this symplectomorphism as the Hamiltonian flow of a *non-smooth* function.

We begin by setting up notation for symplectomorphisms of moduli spaces induced by diffeomorphisms of the underlying surface. Let $(X,\underline{\mu})$ be a marked surface. Let $\mathrm{Diff}^+ \subset \mathrm{Diff}(X)$ denote the group of orientation-preserving diffeomorphisms of X and $\mathrm{Map}^+(X,\underline{\mu})$ the group of isotopy classes of orientation preserving diffeomorphisms ϕ of X preserving the labels $\underline{\mu}$:

$$\mathrm{Map}^+(X,\underline{\mu}) = \{\phi \in \mathrm{Diff}^+(X) \mid \phi(\{x_i\}) = \{x_i\},\ \phi(x_i) = x_j \implies \mu_i = \mu_j\ \forall i,j\}/\sim.$$

The following is elementary and left to the reader:

Proposition 3.4. (Symplectomorphisms associated to mapping class group elements) *Pullback defines a homomorphism from* $\mathrm{Map}^+(X,\underline{\mu})$ *to the group of symplectomorphisms* $\mathrm{Diff}(M(X,\underline{\mu}),\omega)$ *of* $M(X,\underline{\mu})$,

$$\mathrm{Map}^+(X,\underline{\mu}) \to \mathrm{Diff}(M(X,\underline{\mu}),\omega),\quad [\phi] \mapsto ([A] \mapsto [(\phi^{-1})^*A]).$$

In particular, a Dehn twist on the surface induces a symplectomorphism of the moduli space of bundles. Our aim is to describe this symplectomorphism as a Hamiltonian flow of a function relating to the holonomy. The space of connections on the trivial bundle $S^1 \times G \to S^1$ may be canonically identified with the space of $\mathfrak{g}$-valued one-forms

$$L\mathfrak{g}^\vee := \Omega^1(S^1, \mathfrak{g}).$$

Denote $\pi : \mathbb{R} \to S^1$ and let

$$\gamma_{t_0,t_1}(t) = \pi(t_0 + t(t_1 - t_0)), \quad t \in [0,1]$$

denote the standard path from t_0 to t_1. Parallel transport along γ_{t_0,t_1} defines a map

$$\rho_{t_0,t_1} : L\mathfrak{g}^\vee \to G.$$

Given an embedded oriented circle Y in X disjoint from the markings, we suppose that Y is the image of a path

$$Y = \iota([0,1]), \quad \iota : [0,1] \to X, \quad \iota(0) = \iota(1)$$

such that $\iota|(0,1)$ is an embedding. Let $\rho_{0,1}$ denote parallel transport once around Y and define

$$\rho_Y : M(X, \underline{\mu}) \to \mathfrak{A}, \quad [A] \mapsto [\rho_{0,1}(A|Y)]$$

by mapping an equivalence class of flat connections $[A]$ to the conjugacy class of the holonomy of any representative A around Y. The map ρ_Y is independent of the choice of base point on Y. Let

$$h_Y : M(X, \underline{\mu}) \to \mathbb{R}_{\geq 0}, \quad [A] \mapsto (\rho_Y([A]), \rho_Y([A]))/2.$$

The function h_Y is smooth on the inverse image $\rho_Y^{-1}(\mathfrak{A}^\circ)$ of the interior $\mathfrak{A}^\circ$ of the Weyl alcove.

Proposition 3.5. (The action of a Dehn twist is a Hamiltonian flow, c.f. [2, Theorem 4.5]) *Let $(X, \underline{\mu})$ be a marked surface such that $M(X, \underline{\mu})$ contains no reducibles, $Y \subset X$ an embedded circle and $\tau_Y : X \to X$ a Dehn twist around Y. Then τ_Y acts on $M(X, \underline{\mu})$ by the time-one Hamiltonian flow of h_Y on $\rho_Y^{-1}(\mathfrak{A}^\circ)$. In particular, the time-one Hamiltonian flow of h_Y extends smoothly to all of $M(X, \underline{\mu})$.*

The idea of the proof is to describe the twist upstairs on the moduli space of the cut surface, and then descend to the glued surface. Recall from Remark 3.3 that the moduli space $M(X_{\mathrm{cut}}, \underline{\mu})$ has an LG^2-action. In particular the action of LG^2 restricts to an action of the subgroup of constant loops G^2. For $[A] \in M(X_{\mathrm{cut}}, \underline{\mu})$ the notation

$$(1, \rho_{0,1}(A|(\partial X_{\mathrm{cut}})_2))[A] \in M(X_{\mathrm{cut}}, \underline{\mu})$$

indicates the connection obtained by acting by the holonomy $\rho_{0,1}(A|(\partial X_{\mathrm{cut}})_2)$ of the connection on the second component of the boundary $(\partial X_{\mathrm{cut}})_2$.

Lemma 3.6. *The twist τ_Y acts on $M(X_{\mathrm{cut}}, \underline{\mu})$ by changing the framing of A by the holonomy along the second boundary component:*

$$(\tau_Y^{-1})^* : M(X_{\mathrm{cut}}, \underline{\mu}) \to M(X_{\mathrm{cut}}, \underline{\mu}), \quad [A] \mapsto (1, \rho_{0,1}(A|(\partial X_{\mathrm{cut}})_2))[A].$$

Proof. Since a Dehn twist along a circle in a Riemann surface is only defined up to isotopy, we may assume that τ_Y is a Dehn twist along a small translation of the boundary component $(\partial X_{\mathrm{cut}})_2$. This twist induces a Dehn twist on X_{cut}, also denoted τ_Y.

Step 1: We show that on the cut surface X_{cut} the Dehn twist τ_Y maps any connection to a gauge equivalent connection. Suppose first that X_{cut} is connected, and choose a base point $x_0 \in X_{\mathrm{cut}}$. Given a connection A, the connections $A, (\tau_Y^{-1})^* A$ determine representations of the fundamental group

$$(17) \qquad \mathrm{Hol}_A : \pi_1(X_{\mathrm{cut}}, x_0) \to G, \quad \mathrm{Hol}_{(\tau_Y^{-1})^* A} : \pi_1(X_{\mathrm{cut}}, x_0) \to G$$

given by mapping any homotopy class of loops to the holonomy of the connection. The two representations are equal, since the generators of $\pi_1(X_{\mathrm{cut}}, x_0)$ have representatives that are disjoint from the support of τ_Y (or alternatively, since τ_Y is homotopic to the identity). Therefore, $(\tau_Y^{-1})^* A$ is gauge equivalent to A.

Step 2: We compute the gauge transformation relating the two connections above on the boundary. The necessary gauge transformation $g : X \to G$ given by the difference in parallel transports of A and $(\tau_Y^{-1})^* A$. For any $x \in X_{\mathrm{cut}}$, choose a path γ_x from x_0 to x and let $\rho^A(\gamma_x) \in G$ denote parallel transport along γ_x using the connection A, and similarly for the pull-back conection $(\tau_Y^{-1})^* A$. Define a gauge transformation

$$(18) \qquad g : X_{\mathrm{cut}} \to G, \quad x \mapsto \rho^{(\tau_Y^{-1})^* A}(\gamma_x)(\rho^A(\gamma_x))^{-1}.$$

The gauge transformed connection gA has parallel transport along γ_x given by

$$\rho^{(\tau_Y^{-1})^* A)}(\gamma_x)(\rho^A(\gamma_x))^{-1} \rho^A(\gamma_x) = \rho^{(\tau_Y^{-1})^* A)}(\gamma_x).$$

Therefore,

$$gA = (\tau_Y^{-1})^* A.$$

To compute the gauge transformation g, denote by

$$\rho_{0,t} := \rho_{0,t}(A|(\partial X_{\mathrm{cut}})_2)$$

parallel transport from 0 to t along the boundary $(\partial X_{\mathrm{cut}})_2 \cong S^1$. Define a path from x_0 to x_2 on the boundary, identified with $t \in S^1$ by concatenating a path from x_0 to x_1 and a path from 0 to t in S^1. The parallel transports of A resp. $(\tau_Y^{-1})^* A$ are

$$\rho^A(\gamma_{x_2}) = \rho_{0,t} \rho^A(\gamma_{x_1}), \quad \rho_{x_2}^{(\tau_Y^{-1})^* A} = \rho_{0,t} \rho^{(\tau_Y^{-1})^* A}(\gamma_{x_1}) = \rho_{0,t} \rho_{0,1} \rho^A(\gamma_{x_1}).$$

It follows that (18) is given at points t on the second boundary component $(\partial X)_2 \cong S^1$ by

$$(19) \qquad g(t) = \rho_{0,t} \rho_{0,1} g_0 (\rho_{0,t} g_0)^{-1} = \rho_{0,t} \rho_{0,1} \rho_{0,t}^{-1}.$$

Step 3: We identify the gauge transformation on the boundary with the loop group element in the statement of the Lemma. We simplify the formula (19) as follows. After gauge transformation we may assume that the restriction of A to $(\partial X_{\mathrm{cut}})_2)$ is

of the form $\xi_2 d\theta$, for some $\xi_2 \in \mathfrak{g}$. The set of group elements $\rho_{0,t} = \exp(t\xi_2)$ forms a one-parameter subgroup and $\rho_{0,t}$ and $\rho_{0,1}$ commute. Hence

$$g|(\partial X_{\mathrm{cut}})_2 = \rho_{0,1} = \exp(\xi_2), \quad g|(\partial X_{\mathrm{cut}})_1 = 1$$

as claimed. The case that X_{cut} is disconnected is similar, using that τ_Y is trivial on one of the components. $\qquad\square$

Proof of Proposition 3.5. By Lemma 3.6, the restriction of $(\tau_Y^{-1})^*$ to $M(X_{\mathrm{cut}}, \underline{\mu})^\circ$ of (14) has the form

$$M(X_{\mathrm{cut}}, \underline{\mu})^\circ \to M(X_{\mathrm{cut}}, \underline{\mu})^\circ, \quad [A] \mapsto (1, \exp(\xi_2))[A].$$

Thus the action of τ_Y is given by the time-one Hamiltonian flow of the function

$$\tilde{h}_Y : M(X_{\mathrm{cut}}, \underline{\mu})^\circ \to \mathbb{R}, \quad [A] \mapsto (\xi_2, \xi_2)/2.$$

The Proposition follows since the restriction of $\tilde{h}_Y$ to the zero-level set in $M(X_{\mathrm{cut}}, \underline{\mu})$ is a lift of h_Y. $\qquad\square$

For later use, we recall the following facts about level sets of holonomy maps which follow from the *gluing equals reduction* description in Meinrenken-Woodward [32, Section 3.4]. We recall the involution $* : \mathfrak{A} \to \mathfrak{A}$ from (11). Let X_{cap} denote the surface obtained from X_{cut} capping off with a pair of disks, with an additional marked point on each disk, as in Figure 1.
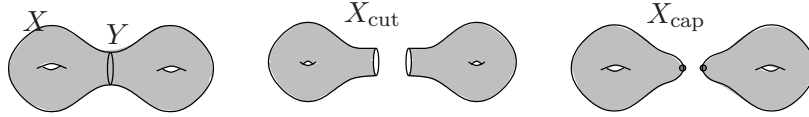

FIGURE 1. Capping off a surface

Lemma 3.7. (Quotients of holonomy level sets) *Let $(X, \underline{\mu})$ be a marked surface, and $Y \subset X$ an embedded circle. For any $\lambda \in \mathfrak{A}$, let $G_{\exp(\lambda)}$ denote the centralizer of $\exp(\lambda)$. The product $G_{\exp(\lambda)} \times G_{\exp(\lambda)}$ acts on $\Phi_1^{-1}(\lambda) \times \Phi_2^{-1}(\lambda)$ with diagonal quotient resp. full quotient*

$$\begin{aligned}
\rho_Y^{-1}(\lambda) &\cong (\Phi_1^{-1}(\lambda) \times \Phi_2^{-1}(\lambda))/G_{\exp(\lambda)} \\
M(X_{\mathrm{cap}}, \underline{\mu}, \lambda, *\lambda) &\cong (\Phi_1^{-1}(\lambda) \times \Phi_2^{-1}(\lambda))/G_{\exp(\lambda)}^2.
\end{aligned}$$

If all points in the level set of (λ, λ) have the same stabilizer $H \subset G_{\exp(\lambda)}^2$ up to conjugacy, then

$$\rho_Y^{-1}(\lambda) \to M(X_{\mathrm{cap}}, \underline{\mu}, \lambda, *\lambda)$$

is a $G_{\exp(\lambda)} \backslash G_{\exp(\lambda)}^2 / H$-fiber bundle and equips $\rho_Y^{-1}(\lambda)$ with the structure of a fibered coisotropic.

3.3. **Full twists for rank two bundles.** In this section we show that Dehn twists on a surface induce fibered Dehn twists of the moduli space of flat rank two bundles with trivial determinant. The following is a slight generalization of a result of M. Callahan (unpublished) resp. P. Seidel [40, Section 1.7] in the case of a separating resp. non-separating curve on a surface.

Theorem 3.8. (For $G = SU(2)$, Dehn twists on the surface act by fibered Dehn twists on the moduli space) *Suppose $G = SU(2)$, and $(X, \underline{\mu})$ is a marked surface such that $M(X, \underline{\mu})$ contains no reducibles. Let $Y \subset X$ be an embedded circle.*

(a) *(Separating case gives a codimension one Dehn twist) If Y is separating then τ_Y acts on $M(X, \underline{\mu})$ by a fibered Dehn twist along the fibered coisotropic*

$$C_\lambda := \rho_Y^{-1}(\lambda)$$

for any $\lambda \in (0, \frac{1}{2})$ such that $M(X, \underline{\mu}, \lambda, *\lambda)$ contains no reducibles.

(b) *(Non-separating case gives a codimension three Dehn twist) If Y is non-separating and $M(X_{\mathrm{cap}}, \underline{\mu})$ contains no reducibles, then τ_Y acts by a fibered Dehn twist along the fibered coisotropic*

$$C_{1/2} := \rho_Y^{-1}(1/2)$$

of bundles with holonomy along Y equal to $\exp(1/2) = -I \in SU(2)$.

Proof. We have already expressed the action of the Dehn twist as the action of holonomy in Proposition 3.5. It remains to identify these flows as Dehn twists in the rank two case. For this, it suffices as in Theorem 2.12 to show that the corresponding Dehn twists induce equivariant Dehn twists on the moduli space of the cut surface.

Step 1: we compute the generic stabilizers, in order to establish that the level sets in the statement are fibrations, and examine the action of the Dehn twists on the level sets. By the irreducible-free assumption, the generic stabilizer of the LG^2 action on $M(X_{\mathrm{cut}}, \underline{\mu})$ is canonically identified with $Z^{\#\pi_0(X_{\mathrm{cut}})}$, where Z is the center of G and $\#\pi_0(X_{\mathrm{cut}})$ is the number of components; the identification is via evaluation at any point on the boundary. By Lemma 3.7 the level set $\rho_Y^{-1}(\lambda)$ with $\lambda \in (0, 1/2)$ resp. $\lambda = 1/2$ is a fibered coisotropic with fiber

$$G_{\exp(\lambda)} \backslash G_{\exp(\lambda)}^2 / H = U(1) \backslash U(1)^2 / (\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S^1$$

in the non-separating case and

$$G_{\exp(\lambda)} \backslash G_{\exp(\lambda)}^2 / H = SU(2) \backslash SU(2)^2 / \mathbb{Z}_2 \cong S^3$$

in the separating case. Explicitly, the latter isomorphism is given by

$$SU(2) \backslash SU(2)^2 / \mathbb{Z}_2 \to SU(2) \cong S^3, \quad [a, b] \mapsto a^{-1}b.$$

Step 2: We establish the Theorem in the separating case. We write the moduli space as a finite dimensional quotient in a neighborhood of the fibered coisotropic in the statement. Denote the subset with generic holonomy

$$M(X, \underline{\mu})^\circ = M(X_{\mathrm{cut}}, \underline{\mu})^\circ /\!/ T = \rho_Y^{-1}(\mathfrak{A}^\circ).$$

Since $M(X_{\mathrm{cut}}, \underline{\mu})^\circ$ has a $T \times T$-action, there is a residual T-action on $M(X, \underline{\mu})^\circ$ with generic stabilizer $Z^{\#\pi_0(X_{\mathrm{cut}})-1}$. The action of τ_Y on $M(X, \underline{\mu})^\circ$ is Hamiltonian isotopic to a symplectomorphism given by Proposition 3.5 as the Hamiltonian flow of $\rho_Y^2/2$ on the complement of $\rho_Y^{-1}(0), \rho_Y^{-1}(1/2)$.

We use a small Hamiltonian deformation to construct a symplectomorphism that is supported on a neighborhood of a fibered coisotropic. Let $\sigma = \{1/2\} \subset \mathfrak{A}$ denote the endpoint of the alcove. By (16), a neighborhood of $\rho_Y^{-1}(1/2)$ is the symplectic reduction of a neighborhood of a cross-section for $\Phi_1^{-1}(1/2) \times \Phi_1^{-1}(1/2) \subset M(X_{\mathrm{cut}}, \underline{\mu})$ by a diagonal G action. More precisely, evaluation a base points defines an isomorphism $LG^2_{(1/2,1/2)} \cong G^2$. The maximal slice at $(1/2, 1/2)$ is

$$\mathfrak{A}^2_\sigma = LG^2_{(1/2,1/2)}(0, 1/2]^2 \subset L\mathfrak{g}^\vee_{(1/2,1/2)}.$$

Then the quotient of $M(X_{\mathrm{cut}}, \underline{\mu})_\sigma = (\Phi_1 \times \Phi_2)^{-1}(\mathfrak{A}^2_\sigma)$ by $LG_\sigma \cong G$ is a neighborhood of $\rho_Y^{-1}(1/2)$. The function $\Phi_2^2/2 - (\Phi_2/2 - 1/8)$ on $M(X_{\mathrm{cut}}, \underline{\mu})^\circ$ is equal to the restriction of the function $\|\Phi_2 - 1/2\|^2/2$ on $M(X_{\mathrm{cut}}, \underline{\mu})_{1/2}$ which is smooth. Let

$$\beta \in C^\infty(M(X, \underline{\mu})), \quad \beta|_{\rho_Y^{-1}(1/2)} = 1$$

be an cutoff function equal to 1 near $\rho_Y^{-1}(1/2)$ and supported in a small neighborhood of $\rho_Y^{-1}(1/2)$. The function

$$(20) \qquad -\beta(\rho_Y^2/2 - (\rho_Y/2 - 1/8)) \in C^\infty(M(X, \underline{\mu})^\circ)$$

has as smooth extension to $M(X, \underline{\mu})$, which agrees with the action of $(\tau_Y^{-1})^*$ near $\rho_Y^{-1}(1/2)$ since the time-one flow of $\rho_Y/2 - 1/8$ is the identity. Composing the action of $(\tau_Y^{-1})^*$ with this flow yields an equivariant symplectomorphism that is the identity near $\rho_Y^{-1}(1/2)$, and is Hamiltonian isotopic to the original symplectomorphism. A similar modification produces a equivariant Hamiltonian perturbation of $(\tau_Y^{-1})^*$ that is the identity near $\Phi_2^{-1}(0)$.

It now follows from the results on equivariant Dehn twists that the Dehn twist on the surface acts by a fibered Dehn twist on the moduli space. That is, $(\tau_Y^{-1})^*$ is a Dehn twist along any $\Phi_2^{-1}(\lambda) \subset M(X_{\mathrm{cut}}, \underline{\mu})^\circ$ for λ generic. By Proposition 2.15, $(\tau_Y^{-1})^*$ acts as an equivariant Dehn twist on $M(X_{\mathrm{cut}}, \underline{\mu})^\circ$. By Theorem 2.12 the action of $(\tau_Y^{-1})^*$ descends to a fibered Dehn twist on $M(X, \underline{\mu})$.

Step 3: We establish the Theorem in the non-separating case. As before, a neighborhood of the coisotropic is given as a symplectic reduction of a finite dimensional manifold. The level set

$$\widetilde{C}_{1/2} = \Phi_2^{-1}(1/2) \subset U \subset M(X_{\mathrm{cut}}, \underline{\mu})$$

is an equivariant fibered coisotropic in the finite dimensional manifold U since it is the zero level set of a moment map for a free action. Lemma 3.6 shows that $(\tau_Y^{-1})^*$ is an equivariant fibered Dehn twist around $\widetilde{C}_{1/2}$. Again a Hamiltonian perturbation gives a symplectomorphism that acts by the identity on a neighborhood of $\rho_Y^{-1}(0)$.

It follows from Theorem 2.12 that $(\tau_Y^{-1})^*$ acts on $M(X, \underline{\mu})$ by a fibered Dehn twist around $C_{1/2} = \widetilde{C}_{1/2}/\!/G$. $\qquad\qquad\square$

3.4. **Half twists for rank two bundles with fixed holonomies.** In this section we show that a half-twist on a marked surface $(X, \underline{\mu})$ induces a fibered Dehn twist on the moduli space $M(X, \underline{\mu})$ of flat $G = SU(2)$ bundles with fixed holonomy.

Definition 3.9. (Half-twist and corresponding coisotropic) Given a pair of points $x_i, x_j \in X$ with the same label $\mu_i = \mu_j$, let Y denote a circle around x_i, x_j and

$$\tau_Y : X \to X, \quad x_i \mapsto x_j, \quad x_j \mapsto x_i$$

a half-twist along Y that interchanges x_i, x_j. Denote the coisotropic of bundles with trivial holonomy along Y

$$C_Y := \{[A] \in M(X, \underline{\mu}), \ \rho_Y([A]) = 0\}.$$

Example 3.10. (Moduli spaces of flat bundles on the sphere punctured five times) Let $X = S^2$ with five markings $x_1, \ldots, x_5$ all with labels $1/4 \in \mathfrak{A}$. The moduli space of flat bundles

$$M(X, \underline{\mu}) = \left\{ (g_1, \ldots, g_5) \in \mathcal{C}_{1/4}^5 \, | \, g_1 \ldots g_5 = 1 \right\} / G$$

is real dimension four. The real manifold $M(X, \underline{\mu})$ admits the structure of a del Pezzo surface obtained by blowing up the projective plane at four points. This fact follows from the existence of a Kähler structure by the Mehta-Seshadri theorem [31], and computing its homology by any number of standard techniques and noting the rationality of the moduli space or by the more detailed discussion in [25]. The submanifold C_Y from (3.9) given by a loop Y around the i-th and j-th marking is a Lagrangian sphere described as bundles whose holonomy along a loop containing the i-th and j-th markings is the identity:

$$C_Y = \{g_i g_j = 1\} \subset M(X, \underline{\mu}).$$

The intersection diagram of these Lagrangians reproduces the root system A_4 corresponding to the fifth del Pezzo, discussed in general in Manin [27]. After choosing suitable generators for the fundamental group we may assume that i, j are adjacent. The moduli space $M(X, \underline{\mu})$ can be described as the moduli space of closed spherical polygons in $S^3 \cong SU(2)$ with side lengths π and vertices

$$1, \ g_1, \ g_1 g_2, \ldots, \ g_1 g_2 g_3 g_4 g_5 \ = \ 1.$$

The submanifold C_Y consists of closed spherical polygons of side lengths π such that the i and j-th edges are opposite. That is, the polygon consists of a bigon and a triangle as in Figure 2. Let $Y_{i(i+1)}, Y_{(i-1)i} \subset X$ be small circles around the pairs consisting of $i, i+1$ resp. $i-1, i$-th markings. The intersection $C_{Y_{i(i+1)}}, C_{Y_{(i-1)i}}$ is the configuration where the bigon is conincident with the $i + 1$-st edge of the triangle as in Figure 3. See [49, Section 5] for more discussion. More generally, in the case of $2n + 1$ markings, the coisotropics C_{ij} consist of configurations composed of a bigon and an $2n - 1$-gon; each is S^2-fibered over the moduli space with $2n - 1$ markings by forgetting the bigon. Applied to the case $X = S^2$ with five markings,
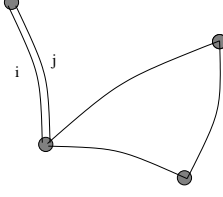
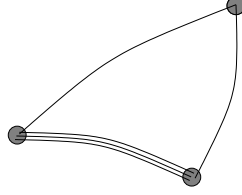FIGURE 2. Coisotropic of degenerate configurations

FIGURE 3. Configuration in the intersection of two coisotropics

this construction gives a lift of generators of the action of the Weyl group S_4 (the symmetric group on five letters) of type A_4 on homology to Dehn twists on the fifth del Pezzo. A lift of the action to the braid group is discussed in Seidel [39].

The following is slight generalization of a result of Seidel [40].

Theorem 3.11. (For $G = SU(2)$ half-twists around pairs of markings act by codimension two fibered Dehn twists on the moduli space) *Let $G = SU(2)$. Let $(X, \underline{\mu})$ be a marked surface such that $M(X, \underline{\mu})$ contains no reducibles and $\mu_i = \mu_j = 1/4$. Let $Y \subset X$ be an embedded circle disjoint from the markings that is the boundary of a disk containing x_i, x_j. Then the set C_Y from (3.9) is a spherically fibered coisotropic submanifold of codimension 2, and the action of τ_Y on $M(X, \underline{\mu})$ is a fibered Dehn twist along C_Y.*

Proof. First show that $M(X, \underline{\mu} - \{\mu_i, \mu_j\})$ is smooth, or equivalently, contains no reducible representations. Indeed, note that the stabilizers of

$$(g_i, g_j), \quad g_i g_j = 1, \quad g_i, g_j \in \mathcal{C}_{1/4}$$

range over all one-parameter subgroups of $SU(2)$. Hence if some point in $M(X, \underline{\mu} - \{\mu_i, \mu_j\})$ has a one-parameter group of automorphisms then there is also a point in $M(X, \underline{\mu})$ with a one-parameter group of automorphisms. Since $M(X, \underline{\mu})$ contains no reducibles, $M(X, \underline{\mu} - \{\mu_i, \mu_j\})$ also contains no reducibles.

By Lemma 3.7, the constancy of generic stabilizer implies the existence of a fibered coisotropic. Let

$$X_{\mathrm{cut}} = X_{\mathrm{cut},1} \sqcup X_{\mathrm{cut},2}, \quad \{x_i, x_j\} \subset X_{\mathrm{cut},2}$$

denote the surface obtained by cutting X into two components $X_{\mathrm{cut},1}, X_{\mathrm{cut},2}$ along Y, so that $X_{\mathrm{cut},2}$ contains the markings x_i, x_j. The generic stabilizer H from (3.9) is the product of stabilizers for the two factors, since Y is disconnecting: The center $Z = \mathbb{Z}_2$ for $M(X_{\mathrm{cut},1}, \underline{\mu} - \{\mu_i, \mu_j\})$ by the previous paragraph, and the maximal

torus $U(1)$ for the right factor $M(X_{\mathrm{cut},2}, \mu_i, \mu_j)$. By 3.7 $C_Y \subset M(X, \underline{\mu})$ from (3.9) is a fibered coisotropic of codimension 2 with fiber

$$G_{\exp(\lambda)} \backslash G^2_{\exp(\lambda)} / H = SU(2)/(\mathbb{Z}_2 \times U(1)) \cong S^2.$$

To describe the action of the half-twist, we first describe the action of the half-twist on the moduli space of the cut surface. By restriction an element of $M(X_{\mathrm{cut}}, \underline{\mu})$ gives a flat connection on the twice-punctured disk $X_{\mathrm{cut},2} - \{x_i, x_j\}$, with holonomies g_i, g_j around the punctures. We choose convenient representatives for g_i, g_j as follows. Let

$$n = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \in N(T)$$

be a representative of the non-trivial element w of the Weyl group $W \cong \mathbb{Z}_2$. The inverse is

$$n^{-1} = -n \in \mathcal{C}_{1/4}.$$

Furthermore, the stabilizer G_n acts on $\mathcal{C}_{1/4}$ by rotation fixing n and n^{-1}. The set nT is diffeomorphic to a circle passing through n. Thus after conjugating g_i, g_j by some $g \in G$, we may assume

$$g_i = n, \quad g_j = tn$$

for some $t \in T$. We compute the action of the half-twist on the holonomies: For the i-th holonomy

$$g_i \mapsto g_i g_j g_i^{-1} = nt = t^{-1}n = \mathrm{Ad}((t^{-1})^{\frac{1}{2}})g_i = \mathrm{Ad}((-g_i g_j)^{\frac{1}{2}})g_i$$

(the square root is unique up to an element of the center, which acts trivially). On the other hand, for the j-th holonomy

$$g_j \mapsto g_i = n = \mathrm{Ad}((t^{-1})^{\frac{1}{2}})tn = \mathrm{Ad}((-g_i g_j)^{\frac{1}{2}})g_j.$$

This shows that τ_Y acts on g_i, g_j by conjugation by $\mathrm{Ad}((-g_i g_j)^{\frac{1}{2}})$. As in the case of a full twist, this implies that $(\tau_Y^{-1})^* A$ is gauge equivalent to the connection obtained from A by acting by $(1, (-g_i g_j)^{\frac{1}{2}})$.

We write a neighborhood of the fibered coisotropic as a finite-dimensional symplectic reduction. Let $\sigma = \{0\}$ denote the endpoint of the alcove. By (15)

$$M(X, \underline{\mu})^\sigma = M(X_{\mathrm{cut}}, \underline{\mu})^\sigma /\!/ SU(2)$$

is a neighborhood of C_Y. Restricted to $M(X_{\mathrm{cut}}, \underline{\mu})^\sigma$, $|\Phi_2|/2$ gives a real-valued function whose Hamiltonian flow is the action of $-I$. Hence $|\Phi_2|/4$ has Hamiltonian flow given by the action of $(-I)^{1/2} = \exp(1/4)$. Similarly $|\Phi_2|^2/4$ has Hamiltonian flow given by the action of $(g_i g_j)^{1/2}$. Combining these remarks shows that the action of τ_Y on $M(X_{\mathrm{cut}}, \underline{\mu})^\sigma$ is the time-one Hamiltonian flow of $|\Phi_2|(|\Phi_2| + 1)/4$, and the level set $\Phi_2^{-1}(0)$ is an equivariant fibered coisotropic. By using a cutoff function as in (20), the action of τ_Y is Hamiltonian isotopic to an equivariant Dehn twist along $\Phi_2^{-1}(0)$. The proof is completed by applying Theorem 2.12. $\square$

3.5. **Half-twists for higher rank bundles.** For special labels, a half twist on the marked surface X induces a fibered Dehn twist on the moduli space of flat $SU(r)$ bundles with fixed holonomy.

Definition 3.12. (Khovanov-Rozansky modification on moduli spaces) Let $r \geq 2$ and let $\omega_k \in \mathfrak{A}$, $k = 0, \ldots, r - 1$ denote the vertices of the alcove. We identify $\mathfrak{g}$ with $\mathfrak{g}^\vee$ using the basic inner product for which the roots have norm-square equal to 2. Under this identification $\omega_0 = 0$ while

$$\omega_k = (\underbrace{(r - k)/r, \ldots, (r - k)/r}_{k}, \underbrace{-k/r, \ldots, -k/r}_{r-k}), \quad 1 \leq k \leq r$$

are the fundamental weights of $G = SU(r)$. Let $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ be a collection of labels with

$$\mu_i = \mu_j = \nu_k^1 := (\omega_k + \omega_{k+1})/2$$

for some i, j, k. (That is, μ_i, μ_j equal the midpoint ν_k^1 of the edge of $\mathfrak{A}$ connecting the k-th vertex with the $k + 1$-st vertex.) Let C_Y be the subset of $M(X, \underline{\mu})$ defined by

$$(21) \qquad\qquad C_Y = \{[A] \in M(X, \underline{\mu}) \,|\, \rho_Y([A]) \sim \exp(\nu_k^2)\}$$

where ν_k^2 is the midpoint between the vertices ω_k, ω_{k+2} of the alcove $\mathfrak{A}$,

$$\nu_k^2 := (\omega_{k+2} + \omega_k)/2.$$

For example,

$$(r = 2, \quad k = 0) \quad \Longrightarrow \quad (\nu_k^1 = 1/4, \quad \nu_k^2 = 0)$$

while

$$(r = 3, \quad k = 0) \quad \Longrightarrow \quad (\nu_k^1 = \omega_1/2, \quad \nu_k^2 = \omega_2/2).$$

Denote by

$$M(X, \underline{\mu}, (\mu_i, \mu_j) \mapsto \nu_k^2) := M(X, \mu_1, \ldots, \hat{\mu}_i, \ldots, \hat{\mu}_i, \ldots, \mu_n, \nu_k^2)$$

the moduli space of flat bundles on X with labels obtained by removing $\mu_i, \mu_j = \nu_k^1$ and adding ν_k^2.

Theorem 3.13. (Codimension two fibered Dehn twists via half-twists) *Suppose that $G = SU(r)$ and $\mu_i, \mu_j, \nu_k^1, \nu_k^2$ are as in Definition 3.12 such that the moduli spaces $M(X, \underline{\mu})$ and $M(X, \underline{\mu}, (\mu_i, \mu_j) \mapsto \nu_k^2)$ contain no reducibles. Let $Y \subset X$ denote an embedded circle enclosing only the i-th and j-th markings. Then the subset C_Y from (21) is a spherically fibered coisotropic submanifold of codimension 2, fibered over the moduli space $M(X, \underline{\mu}, (\mu_i, \mu_j) \mapsto \nu_k^2)$. The action of τ_Y on $M(X, \underline{\mu})$ is a fibered Dehn twist along C_Y.*

Proof. First we show that the given level set is a fibered coisotropic. Let g_i resp. g_j denote the holonomies around the i-th resp. j-th marking. The half-twist produces a connection with holonomies in which g_i, g_j have been replaced with $g_i g_j g_i^{-1}, g_i$. Let $\mathcal{C}_1$ denote the conjugacy class of $\exp(\omega_1/2)$. It suffices to consider the case that $k = 0$ so that $g_i, g_j \in \mathcal{C}_1$. That is, let

$$\nu_k^1 = \mu_i = \mu_j = \omega_1/2, \quad \nu_k^2 = \omega_2/2.$$

We begin by choosing convenient representatives of g_i, g_j. We may assume

$$
\begin{aligned}
g_i &= \exp(\omega_1/2) = \exp(2\pi i(\operatorname{diag}((r-1)/2r, -1/2r, \ldots, -1/2r))) \\
&= \operatorname{diag}(-\exp(\pi i/r), \exp(\pi i/r), \ldots, \exp(\pi i/r)).
\end{aligned}
$$

The centralizer of g_i is therefore

$$
Z_{g_i} = S(U(1) \times U(r-1)) \cong SU(r-1).
$$

Let

$$
O = \operatorname{Im}(SO(2, \mathbb{R}) \to G, \ A \mapsto \operatorname{diag}(A, 1, 1, 1 \ldots, 1))
$$

denote the subgroup of real orthogonal rotations in the first two coordinates in $\mathbb{C}^r$. Since g_i is the product of $\operatorname{diag}(-1, 1 \ldots, 1)$ with a central element in $U(r)$, the adjoint action of g_i on O is given by $g_i o g_i^{-1} = o^{-1}, o \in O$. So

$$
o g_i = \operatorname{Ad}(o^{1/2}) g_i \in \mathcal{C}_1, \forall o \in O.
$$

The conjugacy class $\mathcal{C}_1$ is a symmetric space of rank one. In particular Z_{g_i} acts transitively on the unit sphere in $T_{g_i}\mathcal{C}_1$. This implies that the map $O g_i \to \mathcal{C}_1/Z_{g_i}$ is surjective. By conjugating g_i, g_j by an element of Z_{g_i} we may choose the second element g_j so that

$$
g_j = o g_i = g_i o^{-1}, \quad \text{for some } o \in O.
$$

The subgroup O is conjugate to the one-parameter subgroup generated by the simple root α_1 (or rather its dual coweight). It follows that the square of $\mathcal{C}_1$ in G is given by the union of conjugacy classes

(22) $$ \mathcal{C}_1^2 = \operatorname{Ad}(G)\{g_1^2 o \mid o \in O\} = \bigcup_{\epsilon \in [0, -\frac{1}{2}]} \mathcal{C}_{\omega_1 + \epsilon \alpha_1}, $$

In particular, the conjugacy class $\mathcal{C}_2$ of $\exp(\omega_2/2)$ appears in $\mathcal{C}_1^2$ as the orbit of the element

$$
\exp(\omega_1/2) \exp(s_1 \omega_1/2) = \exp(\omega_2/2)
$$

where $s_1 \in G$ is a representative first simple reflection. The generic stabilizer for the action of G on $\mathcal{C}_1 \times \mathcal{C}_1$ at the inverse image of $\mathcal{C}_2$ is the maximal torus

$$
S(U(2) \times U(r-2)) \cap \operatorname{Ad}(s_1)(SU(2) \times U(r-2)) = S(U(1) \times U(1) \times U(r-2)).
$$

By Lemma 3.7, C_Y is a fibered coisotropic with fiber

$$
S(U(2) \times U(r-2))/S(U(1) \times U(1) \times U(r-2)) \cong S^2.
$$

Next we identify the action of the half-twist. On the holonomies the half-twist acts by

(23) $$ g_i \mapsto g_i g_j g_i^{-1} = g_i o = \operatorname{Ad}(o^{1/2}) g_i, \quad g_j \mapsto g_i = o g_j = \operatorname{Ad}(o^{1/2}) g_j. $$

Let σ denote the face of $\mathfrak{A}$ containing $\omega_2/2$ and LG_σ the stabilizers so that

$$
M(X, \underline{\mu})^\sigma = M(X_{\text{cut}}, \underline{\mu})^\sigma /\!/ LG_\sigma
$$

is an open neighborhood of C_Y in $M(X, \underline{\mu})$. Let A be a framed connection on X_{cut}. The pull-back connection $(\tau_Y^{-1})^* A$ has the same holonomies as that of $(1, o^{1/2})A$,

where $(1, o^{1/2}) \in LG^2$ denotes the constant element of the loop group with values $1, o^{1/2}$. As in the previous cases, this equality implies that the connections $A, (\tau_Y^{-1})^*A$ are gauge equivalent.

To identify the gauge transformation, suppose that $A|(\partial X)_{\text{cut},2} = \xi d\theta$ for some $\xi \in \mathfrak{g}$. The evaluation of the moment map $\Phi_2([A]) = \xi$ is related to the holonomies around x_i, x_j by $\exp(\xi) = g_i g_j$, since $\exp(\xi)$ is the holonomy around the second boundary circle $(\partial X_{\text{cut}})_2$. Since $g_i^2 = \exp(\omega_1)$ we have

$$o = g_i^{-1}g_j = (g_i^{-2})g_ig_j = \exp(-\omega_1 + \xi).$$

By (22), the image of $M(X_{\text{cut}}, \underline{\mu})^\circ$ under Φ_2 is the interval with endpoints $\omega_1, \omega_2/2$, see Figure 4 for the $SU(3)$ case. It follows that the action of the half-twist is the Hamiltonian flow of the function on $M(X_{\text{cut}}, \underline{\mu})$ whose restriction to $M(X_{\text{cut}}, \underline{\mu})^\circ$ is $(\Phi_2, \alpha_1)^2/2 - (\Phi_2, \omega_1/2)$. An argument using cutoff functions as in (20) shows that the half-twist is an equivariant fibered Dehn twist along $\Phi_2^{-1}(\omega_2/2)$. The claim follows from Theorem 2.12. $\qquad\square$
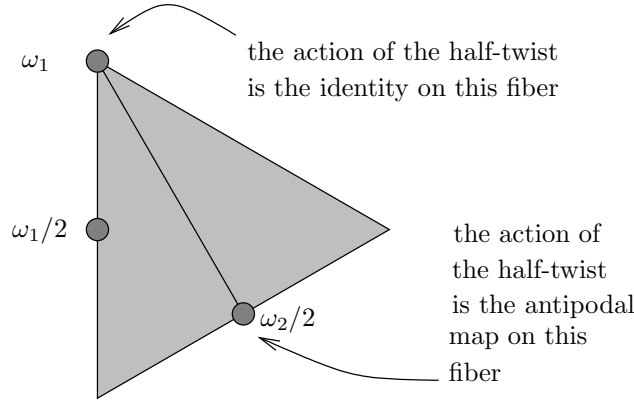

FIGURE 4. Action of half-twist and and product of holonomies

4. PSEUDOHOLOMORPHIC SECTIONS OF LEFSCHETZ-BOTT FIBRATIONS

In this section we describe the relative invariants associated to Lefschetz-Bott fibrations. These are maps between Lagrangian Floer cohomology groups obtained by counting pseudoholomorphic sections.

4.1. **Monotone Lagrangian Floer cohomology.** Novikov rings are not needed to define Lagrangian Floer cohomology for a monotone *pair* of Lagrangian submanifolds. However, we will need our cochain complexes to admit "action filtrations". For this we find it convenient to use the version incorporating a formal variable keeping track of area, as in the construction of the spectral sequence in Fukaya-Oh-Ono-Ohta [13].

Definition 4.1. (Monotonicity conditions)

(a) (Symplectic backgrounds) Fix a monotonicity constant $\lambda \geq 0$ and an even integer $N > 0$. A *symplectic background* is a tuple $(M, \omega, b, \operatorname{Lag}^N(M))$ as follows.
   (i) (Bounded geometry) (M, ω) is a compact smooth symplectic manifold with either empty or convex boundary;
   (ii) (Monotonicity) ω is λ-monotone, i.e. $[\omega] = \lambda c_1(TM)$;
   (iii) (Background class) $b \in H^2(M, \mathbb{Z}_2)$ is a *background class*, which will be used for the construction of orientations; and
   (iv) (Maslov cover) $\operatorname{Lag}^N(M) \to \operatorname{Lag}(M)$ is an N-fold Maslov cover such that the induced 2-fold Maslov covering $\operatorname{Lag}^2(M)$ is the oriented double cover.
   We often refer to a symplectic background $(M, \omega, b, \operatorname{Lag}^N(M))$ as M.

(b) (Monotone Lagrangians) A Lagrangian $L \subset M \backslash \partial M$ is *monotone* if there is an area-index relation for disks with boundary in L, that is,

$$A(u) = \frac{\lambda}{2} I(u), \quad \forall u : (D, \partial D) \to (M, L)$$

where $A(u) = \langle [\omega], [u] \rangle$ (the pairing of $[\omega] \in H^2(M, L)$ with $[u] \in H_2(M, L)$) is again the symplectic area and $I(u)$ is the Maslov index of u as in [30, Appendix].

(c) (Monotone tuples of Lagrangians) A tuple $(L_b)_{b \in \mathcal{B}}$ of Lagrangians in M is *monotone* if the following holds: Let S be any compact surface with boundary given as a disjoint union of one-manifolds C_b

$$\partial S = \sqcup_{b \in \mathcal{B}} C_b$$

(with C_b possibly empty or disconnected). Then for some constant $c(S, M, (L_b)_{b \in \mathcal{B}})$ independent of u,

(24) $$A(u) = \frac{\lambda}{2} \cdot I(u) + c(S, M, (L_b)_{b \in \mathcal{B}}), \quad \forall u : (S, (C_b)_{b \in \mathcal{B}}) \to (M, (L_b)_{b \in \mathcal{B}})$$

where $I(u)$ is the sum of the Maslov indices of the totally real subbundles $(u|_{C_b})^* TL_b$ in some fixed trivialization of $u^* TM$. There is a similar definition of monotonicity for tuples of Lagrangian correspondences [47].

(d) (Admissible Lagrangians) As mentioned in Section 1 we say that a compact monotone Lagrangian submanifold L is *admissible* if the image of the fundamental group of L in M is torsion and L has minimal Maslov number at least 3. One may also allow the case that L has minimal Maslov number 2 and disk invariant (number of disks passing through a generic point) 0. However, the Maslov index 2 case is discussed separately in 5.3. Any tuple of admissible, monotone Lagrangians is automatically monotone, by the argument in Oh [33]. This argument involves completing each boundary component of the surface by adding a disk obtained by contracting a loop, possibly after passing to a finite cover. Products of admissible Lagrangians are also automatically admissible. Thus if L^0, L^1 resp C. are admissible Lagrangians in M resp. $M^- \times B$ then the tuples (L^0, L^1), $(L^0 \times C, C^t \times L^1)$ are monotone.

(e) (Generalized Lagrangian correspondences) Let M, M' be symplectic manifolds. A *generalized Lagrangian correspondence* $\underline{L}$ from M to M' consists of

(i) a sequence $N_0, \ldots, N_r$ of any length $r + 1 \geq 2$ of symplectic manifolds with $N_0 = M$ and $N_r = M'$,

(ii) a sequence $L_{01}, \ldots, L_{(r-1)r}$ of compact Lagrangian correspondences with $L_{(j-1)j} \subset N_{j-1}^- \times N_j$ for $j = 1, \ldots, r$.

Let $\underline{L}$ from M to M' and $\underline{L}'$ from M' to M'' be two generalized Lagrangian correspondences. Then we define concatenation

$$\underline{L} \# \underline{L}' := \left(L_{01}, \ldots, L_{(r-1)r}, L'_{01}, \ldots, L'_{(r'-1)r'} \right)$$

as a generalized Lagrangian correspondence from M to M''. Moreover, we define the dual

$$\underline{L}^t := \left(L^t_{(r-1)r}, \ldots, L^t_{01} \right).$$

as a generalized Lagrangian correspondence from M' to M. A generalized Lagrangian is called admissible if each component is.

(f) (Lagrangian branes) Generally speaking a Lagrangian *brane* means a Lagrangian with extra structure sufficient for the definition of Floer cohomology. In particular a *grading* of a Lagrangian submanifold $L \subset M$ is a lift of the canonical section $L \to \mathrm{Lag}(M)$ to $\mathrm{Lag}^N(M)$, as in Seidel [41]. A *brane structure* on a connected Lagrangian L consists of a grading and relative spin structure for the embedding $L \to M$ (equivalent to a trivialization of $w_2(M)$ in the relative chain complex for (M, L), see for example [50]). A *Lagrangian brane* is an oriented Lagrangian submanifold equipped with a brane structure. A *generalized Lagrangian brane* is a generalized Lagrangian correspondence equipped with a brane structure.

We define Floer cochains as formal sums of perturbed intersection points. Let (L^0, L^1) be a compact, monotone pair of Lagrangian branes in M. Choose a Hamiltonian $H \in C^\infty([0, 1] \times M)$ and denote by $\phi_t \in \mathrm{Diff}(M, \omega)$ the time t flow of the Hamiltonian vector field $H^\# \in \mathrm{Map}([0, 1], \mathrm{Vect}(M))$. Choose H satisfying the condition that $\phi_1(L^0)$ intersects L^1 transversally. Define the set of *perturbed intersection points*

$$\mathcal{I}(L^0, L^1) := \left\{ x : [0, 1] \to M \,\middle|\, x(t) = \phi_t(x(0)), \; x(0) \in L^0, \; x(1) \in L^1 \right\}.$$

The gradings on L^0, L^1 induce a *degree map*

$$\mathcal{I}(L^0, L^1) \to \mathbb{Z}_N, \quad x \mapsto |x|$$

given by the Maslov index of a path from the lifts in the Maslov cover [41]. By the assumption on the Maslov cover, the mod 2 degree is determined purely by the orientations. That is, the mod 2 degree is 0 resp. 1 if the two Lagrangians meet positively resp. negatively after perturbation. The generalized intersection points decompose into subsets of intersections points with fixed index:

$$\mathcal{I}(L^0, L^1) = \bigcup_{d \in \mathbb{Z}_N} \mathcal{I}_d(L^0, L^1), \quad \mathcal{I}_d(L^0, L^1) = \{x \in \mathcal{I}(L^0, L^1) \mid |x| = d\}.$$

Denote the space of time-dependent ω-compatible almost complex structures

$$\mathcal{J}_t(M, \omega) := \mathrm{Map}([0,1], \mathcal{J}(M, \omega)).$$

Given

$$J \in \mathcal{J}_t(M, \omega), \quad H \in C^\infty([0,1] \times M)$$

as above, we say that a map $u : \mathbb{R} \times [0,1] \to M$ is (J, H)-holomorphic iff

$$\overline{\partial}_{J,H} u(s,t) := \partial_s u(s,t) + J_{t,u(s,t)}(\partial_t u(s,t) - H_t^\#(u(s,t))) = 0, \quad \forall (s,t) \in \mathbb{R} \times [0,1].$$

For any $x_\pm \in \mathcal{I}(L^0, L^1)$ we denote by $\mathcal{M}(x_-, x_+)$ the space of finite energy (J, H)-holomorphic maps modulo translation in $s \in \mathbb{R}$, and $\mathcal{M}(x_-, x_+)_0$ the subset of formal dimension 0, that is, index 1. The relative spin structures on L^0, L^1 induce a map

$$o : \mathcal{M}(x_-, x_+)_0 \to \{\pm 1\}$$

measuring the difference between the orientation on each element u and the canonical orientation of a point [43, 50]. The Floer cochain complex is the direct sum

$$CF(L^0, L^1) = \bigoplus_{x \in \mathcal{I}(L^0, L^1)} \mathbb{Z}\langle x \rangle.$$

The Floer coboundary operator $\partial : CF(L^0, L^1) \to CF(L^0, L^1)$ is

$$\partial \langle x_- \rangle := \sum_{x_+ \in \mathcal{I}(L^0, L^1)} \left(\sum_{[u] \in \mathcal{M}(x_-, x_+)_0} o(u) \right) \langle x_+ \rangle.$$

For the following, see [33], [13].

Proposition 4.2. (Construction of Floer cohomology) *Let M be a symplectic background, L^0, L^1 admissible Lagrangian branes and H a time-dependent Hamiltonian making the intersection transverse. There exists a comeager subset $\mathcal{J}_t^{\mathrm{reg}}(M) \subset \mathcal{J}_t(M)$ such that every Floer trajectory is regular. In this case, the Floer coboundary ∂ is well-defined and satisfies $\partial^2 = 0$. The* Floer cohomology

$$HF(L^0, L^1) := \ker(\partial)/\mathrm{im}(\partial)$$

is independent of the choice of H and J, up to isomorphism.

We wish for our Floer cochain complexes to admit action filtrations. To obtain these filtrations we pass to versions over Floer cochains over polynomials in a formal variable. Namely let

$$\Lambda = \left\{ \sum_{j=1}^k n_j q^{\nu_j}, \ n_j \in \mathbb{Z}, \ \nu_j \in \mathbb{R} \right\}$$

denote the space of sums of real powers of a formal variable q. In this paper we do not need to complete Λ, that is, the Novikov ring is not needed. The Floer cochain complex over the coefficient ring Λ is the direct sum

$$CF(L^0, L^1; \Lambda) = \bigoplus_{x \in \mathcal{I}(L^0, L^1)} \Lambda\langle x \rangle$$

with coboundary incorporating the energies of trajectories: Let $H_t^\# \in \mathrm{Vect}(M), t \in [0,1]$ be the Hamiltonian vector field of H and

$$\mathcal{E}_H(u) = \int_{\mathbb{R} \times [0,1]} \omega(\partial_s u(s,t), \partial_t u(s,t) - H_t^\#(u(s,t))) \mathrm{d}s\mathrm{d}t$$

the perturbed energy. Then

$$\partial \langle x_- \rangle := \sum_{x_+ \in \mathcal{I}(L^0, L^1)} \left(\sum_{[u] \in \mathcal{M}(x_-, x_+)_0} o(u) q^{\mathcal{E}_H(u)} \right) \langle x_+ \rangle$$

As before, the *Floer cohomology* with Λ coefficients

$$HF(L^0, L^1; \Lambda) := \ker(\partial)/\mathrm{im}(\partial)$$

is independent of the choice of H and J, up to isomorphism of graded Λ-modules. More generally, for any admissible generalized Lagrangian correspondence $\underline{L}$ from a point to a point there is [47] a *quilted Floer cohomology group* $HF(\underline{L}; \Lambda)$ defined by counting quilted strips.

Remark 4.3. (Specialization) In the monotone setting the natural map

$$HF(L^0, L^1; \Lambda)/(q - \mu) \to HF(L^0, L^1)$$

is an isomorphism for any non-zero μ. Indeed, for any generators $x, y \in \mathcal{I}(L^0, L^1)$ let $c(x,y)$ be the constant in the monotonicity relation for strips from x to y as in (24). Choose a base point $x \in \mathcal{I}(L^0, L^1)$ and consider the map of Λ-modules

$$\iota : CF(L^0, L^1) \to CF(L^0, L^1), \quad \langle y \rangle \mapsto q^{c(x,y)/2} \langle y \rangle.$$

Additivity of index and energy implies that

$$c(x,y) + c(y,z) = c(x,z), \quad \forall x, y, z \in \mathcal{I}(L^0, L^1).$$

Hence for any intersection point $y \in \mathcal{I}(L^0, L^1)$,

$$\begin{aligned}
\partial \iota \langle y \rangle &= \sum_{z,[u]} q^{\mathcal{E}_H(u) + c(x,y)/2} \langle z \rangle \\
&= \sum_{z,[u]} q^{\lambda I(u)/2 + c(y,z)/2 + c(x,y)/2} \langle z \rangle \\
&= \sum_{z,[u]} q^{\lambda/2 + c(x,z)/2} \langle z \rangle = q^{\lambda/2} \iota \partial_1 \langle y \rangle
\end{aligned}$$

where ∂_1 is obtained from ∂ by setting $q = 1$. It follows that the coboundary for $HF(L^0, L^1)$ is, up to an automorphism, equal to the coboundary for $HF(L^0, L^1; \Lambda)$ up to a rescaling by a power of q. The claim follows.

For later use we recall a basic fact about approximate intersection points.

Lemma 4.4. *Suppose that M is a compact Riemannian manifold and $L^0, L^1 \subset M$ are compact submanifolds. For any $\epsilon > 0$ there exists an $\delta > 0$ such that if $m \in M$ is a point with $d(m, L^0) < \delta$ and $d(m, L^1) < \delta$, then $d(m, L^0 \cap L^1) < \epsilon$.*

Proof. Suppose otherwise. Then there exists an $\epsilon > 0$, a sequence $\delta_\nu \to 0$ and a sequence $m_\nu \in M$ with

$$d(m, L^0) < \delta_\nu, \quad d(m, L^1) < \delta_\nu, \quad d(m, L^0 \cap L^1) > \epsilon.$$

By compactness of M, after passing to a subsequence we may assume that m_ν converges to a point m, necessarily in L^0 and L^1. Then

$$d(m_\nu, L^0 \cap L^1) \to d(m, L^0 \cap L^1) = 0.$$

This is a contradiction. $\qquad\square$

It follows that any map that is sufficiently close to both L^0 and L^1 at every point in the domain is in fact contained in a small neighborhood of $L^0 \cap L^1$.

4.2. **Relative invariants for Lefschetz-Bott fibrations.** We may now associate to Lefschetz-Bott fibrations over surfaces with strip-like ends relative invariants that are morphisms of Floer cohomology groups associated to the ends, given by counting pseudoholomorphic sections. The following material can also be found, in a slightly different form, in Perutz [35].

Definition 4.5. (Monotonicity for Lefschetz-Bott fibrations) Let S be a surface without boundary or strip-like ends, and $\pi : E \to S$ a symplectic Lefschetz-Bott fibration. Let

$$\Gamma(E) : \{u : S \to E \mid \pi \circ u = \mathrm{Id}_S\}$$

denote the set of smooth sections of E.

(a) Given $u \in \Gamma(E)$ with image disjoint from the critical set define its *index* and *symplectic area*

$$I(u) = 2(c_1(u^* T^{\mathrm{vert}} E), [S]), \quad A(u) = \int_S u^* \omega_E.$$

Note that the form ω_E is only fiber-wise symplectic. Thus the area $A(u)$ may be negative, in the general case.

(b) A symplectic Lefschetz-Bott fibration E is *monotone* with monotonicity constant $\lambda \geq 0$ if there exists a constant $c(E)$ such that

$$\lambda I(u) = 2A(u) + c(E), \quad \forall u \in \Gamma(E).$$

Remark 4.6. (Behavior of the monotonicity constants under shifts) Given a symplectic Lefschetz-Bott fibration $(\pi : E \to S, \omega_E)$ and a non-negative, compactly supported two-form $\omega_S \in \Omega_c^2(S)$, another Lefschetz-Bott fibration may be constructed by replacing ω_E with $\omega_E + \pi^* \omega_S$. The symplectic area of any section changes under this operation by

$$\langle u^*[\omega_E + \pi^* \omega_S], [S] \rangle - \langle u^*[\omega_E], [S] \rangle = \int_S \omega_S.$$

The monotonicity constant therefore changes by $-2 \int_S \omega_S$.

Proposition 4.7. *Let $\pi : E \to S$ be a symplectic Lefschetz-Bott fibration with S, E compact. If the generic fiber M of E is monotone, $H_1(M) = 0$ and all vanishing cycles in E have codimension at least 2 then E is monotone.*

Proof. We first show that the homology classes of any two sections differ by homology classes in the fibers. For each critical value $s_i \in S^{\text{crit}}$ let $\rho_i : M \to E_{s_i}$ denote the map given by symplectic parallel transport. In particular on the vanishing cycles this map collapses the null foliation. We claim that for any two sections $u_0, u_1 \in \Gamma(E)$, the push-forwards $u_{j,*}[S]$ differ by an element in the span of the homology of the fibers $H_2(M), H_2(E_{s_i})$. The Leray-Fáry spectral sequence for the map $\pi : E \to S$, see Fáry [12, Section 10], Bredon [7, Section IV.12] allows the computation of the homology as follows. Equip the base with the filtration

$$\emptyset \subset \{s_0, \dots, s_n\} \subset S.$$

The fibers of E are diffeomorphic over each associated graded piece $\{s_0, \dots, s_n\}, S - \{s_0, \dots, s_n\}$. Denote by $\mathcal{H}_q(M)$ the local system of homology groups with fibers $H_q(\pi^{-1}(z)) \cong H_q(M)$ over $S - \{s_0, \dots, s_n\}$. The second page in the spectral sequence is the direct sum of relative homologies of successive spaces in the filtration with values in the homology of the fiber,

$$\mathcal{C}_{p,q} = H_p(S - \{s_1, \dots, s_n\}; \mathcal{H}_q(M)) \oplus \bigoplus_{i=1}^{n} H_q(E_{s_i}).$$

Since $H_1(M) = 0$, the contribution from $p = q = 1$ vanishes. It follows that degree two homology classes in the total space arise either as homology classes in the generic fibers invariant under monodromy; homology classes in the special fibers; or homology classes of the base. The projection π_* is an isomorphism on the classes of the last type. It follows that the degree 2 part of the kernel of π_* under the projection map is generated by the image of $H_2(M)$ and $H_2(E_{s_i})$.

We now show the monotonicity relation, using the spectral sequence for the parallel transport maps to the special fibers. By the previous paragraph, it suffices to check monotonicity evaluated on homology classes of fibers. For this we show that maps $\rho_{i,*} : H_2(M) \to H_2(E_{s_i})$ are surjective. Consider the long exact sequence

(25) $H_2(M) \to H_2(E_{s_i}) \to H_2(\text{Cone}(\rho_i)) \to H_1(M)$

where $\text{Cone}(\rho_i)$ is the mapping cone on ρ_i. Since ρ_i is a diffeomorphism away from the vanishing cycle, $\text{Cone}(\rho_i)$ admits a deformation retraction to $\text{Cone}(p_i)$ where $p_i : C_i \to B_i$ is the projection. Since $C_i = P_i \times_{SO(c+1)} S^c$, we have

$$\text{Cone}(p_i) = (P_i \times_{SO(c+1)} S^{c+1}) / \sim$$

where $\sim$ is the equivalence relation that collapses the section corresponding to the fixed point $(0, \dots, 0, 1)$. For $c \geq 2$, the spectral sequence for the fibration $P_i \times_{SO(c+1)} S^{c+1} \to B_i$ implies that any degree two homology class arises from a homology class in the base. On the other hand, the base homology classes $H_2(B_i)$ have trivial image in the cohomology $H(\text{Cone}(p_i))$ of the mapping cone. Hence $H_2(\text{Cone}(p_i)) = 0$. The long exact sequence (25) implies that $H_2(M)$ surjects onto $H_2(E_{s_i})$. $\square$

We now wish to allow our Lefschetz-Bott fibrations to have strip-like ends. The monotonicity conditions in this case will be similar, but with additional constants depending on the limits.

Definition 4.8. (Symplectic Lefschetz-Bott fibrations with strip-like ends)

(a) Let S be a complex curve with boundary obtained from a compact curve with boundary $\overline{S}$ be removing points on the boundary $z_1, \ldots, z_n$. A *strip-like end* for the j-th puncture of S is a holomorphic map

$$\epsilon_j : [0, \pm\infty) \times [0, 1] \to S$$

such that $\exp(2\pi((\epsilon_j^{-1})_1 + i(\epsilon_j^{-1})_2))$ is a local holomorphic coordinate on the closure of the image of ϵ_j; the end is called *incoming* resp. *outgoing* if the sign is negative resp. positive. A *collection of strip-like ends* $\mathcal{E}$ is a set of strip-like ends, one for each $j = 1, \ldots, n$. We write $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_+$ the union of the incoming and outgoing ends.

(b) A symplectic *Lefschetz-Bott fibration over a surface with strip-like ends S* with fiber given by a symplectic manifold M is a Lefschetz-Bott fibration $E \to S$ with S^{crit} contained in the interior of S, together with a trivialization

$$\varphi_{S,e} : \epsilon_{S,e}^* E \to (0, \pm\infty) \times [0, 1] \times M$$

for each end $e \in \mathcal{E}$, such that $\varphi_{s,e}^* \omega_E = \pi_M^* \omega_M$ where

$$\pi_M : (0, \pm\infty) \times [0, 1] \times M \to M$$

is projection on the last factor.

(c) Let S be a surface with strip-like ends and $\pi : E \to S$ a symplectic Lefschetz-Bott fibration with fiber M. A *Lagrangian boundary condition* for E is a submanifold $F \subset \partial E$ such that
   (i) $\pi | F$ is a fiber bundle over ∂S;
   (ii) each fiber $F_z \subset E_z, z \in \partial S$ is a Lagrangian submanifold;
   (iii) for each $e \in \mathcal{E}$ there exist Lagrangian submanifolds $L^{0,e}, L^{1,e} \subset M$ such that F is constant sufficiently close to z_e that is,

$$\varphi_{S,e}(F_{\epsilon_{S,e}(s,j)}) = L^{j,e}, \quad \pm s \gg 0;$$

   and
   (iv) for each $e \in \mathcal{E}$, the intersection $L^{0,e} \cap L^{1,e}$ is transversal.

Definition 4.9. (a) (Monotonicity condition for Lefschetz-Bott fibrations with boundary) Let S be a compact surface with boundary, (E, F) a bundle with boundary condition, and $\Gamma(E, F)$ the set of smooth sections $u : (S, \partial S) \to (E, F)$. Each $u \in \Gamma(E, F)$ takes values in the smooth locus of E. By pull-back one obtains bundles

$$u^* T^{\mathrm{vert}} E \to S, \quad (u | \partial S)^* T^{\mathrm{vert}} F \to \partial S.$$

Taking the Maslov index of this pair gives rise to a *Maslov index* map

$$I : \Gamma(E, F) \to \mathbb{Z}.$$

We also define the *symplectic area*

$$A : \Gamma(E, F) \to \mathbb{R}, \quad u \mapsto \int_S u^* \omega_E$$

keeping in mind that the form ω_E is only symplectic on the fibers. A pair (E, F) *monotone with monotonicity constant* λ if the index depends linearly on the area; that is,

$$\lambda I(u) = 2A(u) + c(E, F), \quad \forall u \in \Gamma(E, F) \tag{26}$$

for some constant $c(E, F)$.

(b) (Linearized operator) Suppose that S is a surface with boundary and strip-like ends $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_+$, and (E, F) a bundle with boundary condition. For any collection

$$(x_e \in \mathcal{I}(L^{0,e}, L^{1,e}))_{e \in \mathcal{E}},$$

let

$$\Gamma(E, F; (x_e)_{e \in \mathcal{E}}) = \left\{ u \in \Gamma(E) \;\middle|\; \begin{array}{c} \lim_{s \to \pm\infty} u(\epsilon_e(s, t)) = x_e, \forall e \in \mathcal{E} \\ u|_{\partial S} \in \Gamma(F) \end{array} \right\}$$

denote the space of sections with boundary values in F and asymptotic limits x_e. For sections $u \in \Gamma(E, F)$ let

$$\overline{\partial} u = \frac{1}{2}(\mathrm{d}u + J(u) \circ \mathrm{d}u \circ j) = 0 \tag{27}$$

denote the Cauchy-Riemann equation associated to the pair (j, J). The linearized operator

$$D_u : \Omega^0(u^* T^{\mathrm{vert}} E, u^* T^{\mathrm{vert}} F) \to \Omega^{0,1}(u^* T^{\mathrm{vert}} E), \quad \xi \mapsto \frac{d}{dt}\Big|_{t=0} \Pi_{t\xi}^{-1} \overline{\partial} \exp_u(t\xi) \tag{28}$$

is given by differentiating the Cauchy-Riemann operator along a path $\exp_u(t\xi)$ of geodesic exponentials, and using parallel transport $\Pi_{t\xi}^{-1}$ back to u. The operator D_u is Fredholm since the boundary conditions at infinity are assumed transversal.

(c) (Monotonicity condition for Lefschetz-Bott fibrations with strip-like ends) The pair (E, F) is *monotone* with monotonicity constant $\lambda \geq 0$ if for any $(x_e)_{e \in \mathcal{E}}$ there exists a constant $c(E, F; (x_e)_{e \in \mathcal{E}})$ such that

$$\lambda \operatorname{Ind}(D_u) = 2A(u) + c(E, F; (x_e)_{e \in \mathcal{E}}), \quad \forall u \in \Gamma(E, F; (x_e)_{e \in \mathcal{E}}). \tag{29}$$

Proposition 4.10. (Condition for monotonicity of the fiber to imply monotonicity of a fibration with strip-like ends) *Let $E \to S$ be a symplectic Lefschetz-Bott fibration over a surface with boundary S with connected and simply-connected fibers. Let $F \subset E|\partial S$ be a Lagrangian boundary condition with connected and simply-connected fibers. Suppose that the generic fiber of E is monotone and the vanishing cycles of E have codimension at least 2. Then (E, F) is monotone.*

Proof. First consider the case without ends. As in Proposition 4.7, any two sections of E differ by a homology sphere $H_2(E_s)$ in some fiber E_s of E, and these are generated by classes in the generic fiber by the codimension assumption for which monotonicity holds. In the case with ends, as in [46], one may use gluing to reduce to the case that S has no strip-like ends as in the statement of the Proposition. More precisely, fix a section $v : S \to E$ with limits $\underline{x}_+$. View v as a map from the surface $S^- = (S, -j)$ with complex structure reversed. Then for any other section

$u : S \to E$ with the same limits, gluing along the strip-like ends produces a section of the doubled surface $w : S\#S^- \to E\#E^-$. Then

$$
\begin{aligned}
\mathrm{Ind}(D_u) &= \mathrm{Ind}(D_w) - \mathrm{Ind}(D_v) \\
&= I(w) + \dim(E)\chi(S\#S^-)/2 - \mathrm{Ind}(D_v) \\
&= \lambda A(w) + \dim(E)\chi(S\#S^-)/2 - \mathrm{Ind}(D_v) \\
&= \lambda A(u) + \lambda A(v) + \dim(E)\chi(S\#S^-)/2 - \mathrm{Ind}(D_v).
\end{aligned}
$$

Since the last three terms are independent of u, this proves the statement of the Proposition. $\square$

We now turn to the construction of relative invariants associated to symplectic Lefschetz-Bott fibrations. Let $\pi : E \to S$ be such a fibration, equipped as in Definition 2.1 with a complex structure j_0 on a neighborhood of the critical values in S and J_0 on a neighborhood of the critical points in E. In the case that S has ends, we assume furthermore that almost complex structures $J_e \in \mathcal{J}(M)$ are fixed making the moduli spaces of Floer trajectories for that end regular. We wish to extend these to almost complex structures on the entire fibration and base, so that we may define moduli spaces of pseudoholomorphic sections.

Definition 4.11. (a) (Compatible almost complex structures) Let $\pi : E \to S$ be a Lefschetz-Bott fibration over a surface with strip-like ends S. A complex structure j on S is *compatible* with E if $j = j_0$ in an neighborhood of S^{crit}. An almost complex structure J on E is *compatible* with π, j iff
  (i) $J = J_0$ in a neighborhood of E^{crit};
  (ii) π is (J, j)-holomorphic in a neighborhood of E^{crit}, that is, $J \circ \mathrm{d}\pi = \mathrm{d}\pi \circ j$; and
  (iii) $\omega_E(\cdot, J\cdot)$ is symmetric and positive definite on TE_x^v, for any $x \in E$.
  (iv) J is equal to fixed almost complex structure $j \times J_e, e \in \mathcal{E}(S)$ on the ends of S in the given trivializations on a neighborhood $U_e \subset S$ of each end.
Let $\mathcal{J}(E)$ denote the set of (π, j)-compatible almost complex structures. To see that $\mathcal{J}(E)$ is non-empty, note that the symplectic form defines a connection $\ker(D\pi)^\omega$ on $E \to S$ away from the singular locus, given by the symplectic perpendicular of the vertical part of the tangent space. First choose j equal to j_0 near the critical values. Then take J to equal j on the horizontal subspace, and any almost complex structure compatible with the symplectic form on the vertical subspace.
  (b) (Moduli space of pseudoholomorphic sections) Let $\mathcal{M}(E, F; (x_e)_{e \in \mathcal{E}})$ denote the set of finite area sections $u \in \Gamma(E, F, (x_e)_{e \in \mathcal{E}})$ such that u is (j, J)-holomorphic, with limits $(x_e)_{e \in \mathcal{E}}$ along the ends.

Theorem 4.12. (Existence of regular almost complex structures for symplectic Lefschetz-Bott fibrations) *Suppose that (E, F) is a monotone symplectic Lefschetz-Bott fibration with Lagrangian boundary conditions over a surface S with strip-like ends. There exists a comeager subset $\mathcal{J}^{\mathrm{reg}}(E) \subset \mathcal{J}(E)$ such that*

(a) $\mathcal{M}(E, F; (x_e)_{e \in \mathcal{E}})$ is a smooth manifold with tangent space at u given by $\ker(D_u)$;

(b) the zero-dimensional component $\mathcal{M}(E, F; (x_e)_{e \in \mathcal{E}})_0 \subset \mathcal{M}(E, F; (x_e)_{e \in \mathcal{E}})$ is finite;

(c) the one-dimensional component $\mathcal{M}(E, F; (x_e)_{e \in \mathcal{E}})_1 \subset \mathcal{M}(E, F; (x_e)_{e \in \mathcal{E}})$ has a compactification with boundary

$$\bigcup_{(x_e)_{e \in \mathcal{E}}, f, x'_f} \mathcal{M}(E, F; (x_e)_{e \in \mathcal{E}}, x_f \mapsto x'_f)_0 \times \mathcal{M}(x_f, x'_f)_0$$

consisting of pairs of a section with bubbled-off trajectory; and

(d) any relative spin structure on (E, F) induces a set of orientations on the manifolds $\mathcal{M}(E, F; (x_e)_{e \in \mathcal{E}})_0$ that are coherent in the sense that they are compatible with the gluing maps from (c) in the sense that the inclusion of the boundary in (c) has the signs $(-1)^{\sum_{e<f} |x_e^-|}$ (for incoming trajectories) and $-(-1)^{\sum_{e<f} |x_e^+|}$ (for outgoing trajectories.)

The proof is similar to that of [42] in the exact case. Bubbling for sections can occur only in the fiber. So sphere and disk bubbling on the zero and one-dimensional moduli spaces is ruled out by the monotonicity condition. The construction of coherent orientations is given in [50].

Remark 4.13. (Sketch of construction of orientations from [50]) We briefly recall the construction of orientations: On any disk with Lagrangian boundary conditions where the Lagrangian is equipped with a relative spin structure, the linearized operator D_u over the surface S may be deformed via nodal degeneration to a boundary value problem D'_u on the sphere S_s and a constant boundary value problem on the disk S_d. On the sphere the linearized Cauchy-Riemann operator $D'_u|S_s$ is homotopic to a complex linear operator D''_u. The determinant line $\det(D'_u)$ inherits an orientation via the complex structure on the kernel $\ker(D''_u)$ and cokernel $\operatorname{coker}(D''_u)$. On the other hand, on the disk the boundary condition $(u|_{\partial S_d})^* TL$ admits a canonical stable trivialization determined by the relative spin structure. The determinant line on the disk admits an orientation induced from a trivialization of $(u|_{\partial S_d})^* TL$ and an isomorphism of the kernel $\ker(D'_u|S_d)$ with the tangent space to the Lagrangian $T_{u(z)}L$ at any point on the boundary. These combine to an orientation on the determinant line $\det(D_u) \cong \det(D'_u|S_s) \otimes \det(D'_u|S_d)$.

Remark 4.14. (Brane structures on spherically fibered coisotropics) In order to specify the signs in the exact triangle it is necessary to specify the brane structure on the fibered coisotropic. If C is a spherically fibered coisotropic that reduces to a *spin* principal bundle, then TC admits a relative spin structure for the embedding $C \to M \times B$ with background class $w_2(TM)$ induced by the isomorphism

$$TC \oplus \underline{\mathbb{R}} \cong \pi^* TB \oplus P(TS^c \oplus \underline{\mathbb{R}}) \cong \pi^* TB \oplus P(\underline{\mathbb{R}}^{c+1})$$

where $\underline{\mathbb{R}}$ denotes the trivial bundle with fiber $\mathbb{R}$.

For later use we recall the basic fact that maps with sufficiently small energy must also have small diameter:

Lemma 4.15. *Let $\underline{S}$ be a holomorphic quilt with strip-like ends, with symplectic labels $\underline{M}$ and Lagrangian boundary and seam condition $\underline{L}$. For any $\delta > 0, \ell_0$, there exists $\epsilon > 0$ such that if $u : \underline{S} \to \underline{M}$ is a pseudoholomorphic quilt with energy $E(u) < \epsilon$, and $\gamma : [0,1] \to \underline{S}$ is a path connecting boundary components of length less than ℓ_0, then the length of $u \circ \gamma$ is less than δ.*

Proof. The claim is an application of the mean value inequality for pseudoholomorphic maps [30, Lemma 4.3.1 (i)]: For any compact almost complex manifold M there exist constants $c, \epsilon > 0$ such that for any $r > 0$ and any pseudoholomorphic map $u : B_r(z) \to M$ with energy at most ϵ,

$$(30) \qquad |du(z)|^2 \le \frac{c}{r^2} \int_{B_r(z)} |du(w)|^2 \mathrm{d}w$$

where $B_r(z)$ is a ball of radius r around z. There is a similar version for Lagrangian boundary conditions, in which the right-hand-side is replaced by a half-ball [30, Lemma 4.3.1 (i)]. In our quilted situation, we may apply the mean value inequality in each patch. Since $\underline{S}$ is compact on a complement of the strip-like ends, there exists a constant $r_0 > 0$ such that any point in $\underline{S}$ is contained in ball of radius at least r_0, or half-ball of radius r_0 in $\underline{S}$. By integrating over $\gamma(t)$ we see that the length $\ell(\gamma)$ satisfies

$$
\begin{aligned}
\ell(u \circ \gamma) &= \int_0^1 |\mathrm{d}(u \circ \gamma)(t)))| \mathrm{d}t \\
&\le \int_0^1 \frac{c}{r_0^2} \left(\int_{B_{r_0}(\gamma(t))} |\mathrm{d}u(w)|^2 \mathrm{d}w \right)^{1/2} |\gamma'(t)| \mathrm{d}t \\
&\le cr_0^{-2} E(u)\ell(\gamma) \le cr_0^{-2}\epsilon\ell_0.
\end{aligned}
$$

This proves the claim for ϵ such that $\epsilon < \delta r_0^2 / c\ell_0$. $\qquad\qquad\square$

Definition 4.16. (Relative invariants for Lefschetz-Bott fibrations with strip-like ends) For Lefschetz-Bott fibration E with boundary condition F equipped with a relative spin structure, suppose that the ends $L^{e,0}, L^{e,1}$ of F are transverse and $CF(L^{e,0}, L^{e,1})$ have been defined without Hamiltonian perturbation for each end e. Define

$$C\Phi(E, F; \Lambda) : \bigotimes_{e \in \mathcal{E}_-} CF(L^{e,0}, L^{e,1}; \Lambda) \to \bigotimes_{e \in \mathcal{E}_+} CF(L^{e,0}, L^{e,1}; \Lambda)$$

by

$$(31) \qquad C\Phi(E, F; \Lambda)\left(\otimes_{e \in \mathcal{E}_-} \langle x_e \rangle \right) = \sum_{u \in \mathcal{M}(E, F; (x_e)_{e \in \mathcal{E}})_0} o(u) q^{A(u)} \left(\otimes_{e \in \mathcal{E}_+} \langle x_e \rangle \right)$$

where $o(u) = \pm 1$ are orientations constructed in [50]. By items (c) and (d) of Theorem 4.12, the maps $C\Phi(E, F; \Lambda)$ are cochain maps. Passing to cohomology (and passing to rational coefficients in the case of more than one outgoing end) one obtains a map

$$(32) \qquad \Phi(E, F; \Lambda) : \bigotimes_{e \in \mathcal{E}_-} HF(L^{e,0}, L^{e,1}; \Lambda) \to \bigotimes_{e \in \mathcal{E}_+} HF(L^{e,0}, L^{e,1}; \Lambda)$$

Remark 4.17. (Independence from almost complex structure and fiber-wise symplectic form) The cohomology-level invariants $\Phi(E, F; \Lambda)$ are independent of the choice of compatible almost complex structure J on E, by an argument using parametrized moduli spaces similar to that of Theorem 4.12.

The invariants are independent of the choice of two-form $\omega_E \in \Omega^2(E)$ in the following sense: Given any symplectic Lefschetz-Bott fibration (E, ω_E, π) we may form a new symplectic Lefschetz-Bott fibration $(E, \omega_E + \pi^*\omega_S, \pi)$ by adding on the pull-back of non-negative, compactly supported two-form $\omega_S \in \Omega_c^2(S)$ on the base. Any almost complex structure J_0 compatible with ω_E will also be compatible with $\omega_E + \pi^*\omega_S$, although not necessarily vice-versa. As a result, for any two such almost complex structures $J_k, k \in \{0, 1\}$ compatible with $\omega_E + \lambda_k \pi^*\omega_S$ for some scalars λ_0, λ_1, the moduli spaces

$$\mathcal{M}(E, F, J_0; (x_e)_{e \in \mathcal{E}})_0 \sim \mathcal{M}(E, F, J_1; (x_e)_{e \in \mathcal{E}})_0$$

are cobordant. Since the area of each pseudoholomorphic section changes by the integral $A(S) = \int_S \omega_S$, the invariant $\Phi(E, F; \Lambda)$ is independent of this change up to an overall power $q^{A(S)}$.

The relative invariants of fibrations with "non-negative curvature", in the following sense, have particularly nice properties. Recall the symplectic connection (4) on a Lefschetz fibration $\pi : E \to S$. The spaces TE_e^h have canonical complex structures, induced from the complex structure j on the base S. Define

$$\omega_{E,e}|T^h E = f(e)\pi^*\omega_S$$

for some function $f : E \to \mathbb{R}$.

Definition 4.18. A Lefschetz-Bott fibration E with two-form ω_E has *non-negative curvature* if $\omega_E(v, jv) \geq 0$ for all v in the horizontal subspace TE^h, that is, $f(e) \geq 0$ for all $e \in E$.

Remark 4.19. Non-negative curvature implies that a small perturbation of the two-form is symplectic: Recall that the total space of any Lefschetz-Bott fibration $\pi : E \to S$ admits a canonical isotopy class of symplectic structures given as follows. If $\omega_S \in \Omega^2(S)$ is a sufficiently positive two-form then $\omega_E + \pi^*\omega_S$ is a symplectic form on TE_e for any $e \in E$. If E is compact, then $\omega_E + \pi^*\omega_S$ is symplectic on E for ω_S sufficiently positive. If E is non-negative, then $\omega_E + \pi^*\omega_S$ is symplectic for *any* positive form $\omega_S \in \Omega^2(S)$.

Proposition 4.20. (Non-negative curvature of standard Lefschetz-Bott fibrations) *If $C \subset M$ is a spherically fibered coisotropic, then the standard Lefschetz-Bott fibration E_C of 2.13 has non-negative curvature.*

Proof. Let $v \in V$, the standard representation of $SO(c+1)$, and $(p, v) \in P \times V$. The horizontal subspace $H_v \subset T_v V$ pairs trivially with $\ker(\alpha) \times T(\pi_V^{-1}(\pi_V(v)))$ under the pairing given by the two-form (7), where π_V is the projection (9). It follows that the image $[H_v]$ of H_v in $P(V)$ is the horizontal subspace at $[p, v] \in P(V) := (P \times V)/G$. Let J_V denote the standard complex structure on V, and J_0 the induced complex structure on E. Since J_V is non-negative on H_v, J_0 is non-negative on $[H_v]$. $\square$

Proposition 4.21. *Let E be a Lefschetz-Bott fibration with Lagrangian boundary condition F and relative spin structure. If E has non-negative curvature, then the exponents of q in the formula* (31) *are all non-negative.*

Proof. Since the form $\omega_E(\cdot, J\cdot)$ is non-negative for any $J \in \mathcal{J}(E)$, any pseudoholomorphic section has non-negative area. The q-exponents in (31) are the areas, and so are non-negative as well. $\square$

We do not give formula for the degree of the relative invariant. See [46] for a formula for the degree in the case without singularities.

4.3. **Invariants for quilted Lefschetz-Bott fibrations.** The main difference between the triangle for the fibered case and the original Seidel triangle [42] is the appearance of invariants associated to quilted surfaces. The following definitions are taken from [46].

Definition 4.22. (a) (Quilted surfaces with strip-like ends) A *quilted surface with strip-like ends $\underline{S}$* consists of the following data:

(i) a collection $\underline{S} = (S_k)_{k=1,\dots,m}$ of surfaces with strip-like ends, see [42], [46], called *patches*. Each S_k carries a complex structure j_k and has strip-like ends $(\epsilon_{k,e})_{e\in\mathcal{E}(S_k)}$. Denote the limits of these ends

$$\lim_{s\to\pm\infty} \epsilon_{k,e}(s,t) =: z_{k,e} \in \partial\overline{S}_k$$

and denote the boundary components

$$\partial S_k = (I_{k,b})_{b\in\mathcal{B}(S_k)};$$

(ii) a collection $\mathcal{S}$ of *seams*: pairwise disjoint 2-element subsets

$$\sigma \subset \bigcup_{k=1}^m \bigcup_{b\in\mathcal{B}(S_k)} I_{k,b}$$

and for each $\sigma = \{I_{k,b}, I_{k',b'}\}$ a real analytic isomorphism

$$\varphi_\sigma : I_{k,b} \to I_{k',b'};$$

where the isomorphisms φ_σ should be compatible with the strip-like ends in the sense that on each end φ_σ should be a translation;

(iii) Orderings of the boundary components $\mathcal{B}(S_k), k = 1, \dots, m$ of each patch; and

(iv) Orderings of the incoming and outgoing ends of $\underline{S}$

$$\mathcal{E}_-(\underline{S}) = (\underline{e}_1^-, \dots, \underline{e}_{N_-(\underline{S})}^-), \quad \mathcal{E}_+(\underline{S}) = (\underline{e}_1^+, \dots, \underline{e}_{N_+(\underline{S})}^+).$$

(b) (Quilted symplectic Lefschetz-Bott fibrations) A *(quilted) symplectic Lefschetz-Bott fibration $\underline{E}$* over a quilted surface $\underline{S}$ with strip-like ends consists of a collection of Lefschetz-Bott fibrations $E_k \to S_k, k = 0, \dots, m$. A *Lagrangian boundary/seam condition* for E consists of a collection $\underline{F}$ of submanifolds of the boundaries and seams

$$F_{k,b} \subset E_{k,b}|I_{k,b}, \quad F_{(k_0,b_0),(k_1,b_1)} \subset E_{k_0}|I_{k_0,b_0} \times \varphi_\sigma^*(E_{k_1}|I_{k_1,b_1})$$

where $I_{k,b}$ ranges over true boundary components resp. I_{k_0,b_0}, I_{k_1,b_1} range over identified boundary components, such that

(i) each fiber

$$F_{(k_0,b_0),(k_1,b_1),z} \subset E_{k_0,z} \times (\varphi^*_{(k_0,b_0),(k_1,b_1)} E_{k_1})_z$$

over $z \in \underline{S}$ is a Lagrangian submanifold; and

(ii) over the strip-like ends the the fibers $F_{(k_0,b_0),(k_1,b_1),z}$ over $z \in \underline{S}$ are given by fixed Lagrangians $L^{k_{e,i},b_{e,i}}$ on the strip-like ends, with the property that the composition

$$L^{(k_{e,0},b_{e,0})} \circ L^{(k_{e,1},b_{e,1})} \ldots \circ L^{(k_{e,l(e)},b_{e,l(e)})}$$

is transversal, where $l(e)$ is the number of patches on the end e.

We say that a quilted Lefschetz-Bott fibration is *monotone* if sections satisfy an area-index relation similar that for pseudoholomorphic maps. In the case without singularities (that is, fibrations) admissibility for the Lagrangians guarantees monotonicity in the quilted setting, see [46, Remark 3.7]. For Lefschetz-Bott fibrations, admissibility together with the codimension conditions in Lemma 4.10 guarantee monotonicity, by the same arguments.

Definition 4.23. (Relative invariants for quilted Lefschetz-Bott fibrations) Theorem 4.12 generalizes to the quilted setting: associated to a monotone quilted symplectic Lefschetz-Bott fibration $(\underline{E}, \underline{F})$ we have (ungraded) relative invariants which (working with rational coefficients in the case of more than one outgoing end) maps

$$\text{(33)} \quad \bigotimes_{e \in \mathcal{E}_-} HF(L^{(k_{e,0},b_{e,0})}, L^{(k_{e,1},b_{e,1})}, \ldots, L^{(k_{e,l_-(e)},b_{e,l_-(e)})}; \Lambda)$$

$$\to \bigotimes_{e \in \mathcal{E}_+} HF(L^{(k_{e,0},b_{e,0})}, L^{(k_{e,1},b_{e,1})}, \ldots, L^{(k_{e,l_+(e)},b_{e,l_+(e)})}; \Lambda).$$

These invariants satisfy a composition relation for gluing along strip-like ends by a gluing argument spelled out in [46]. That is, if $\underline{E}, \underline{F}$ is a quilted Lefschetz-Bott fibration obtained from fibrations $\underline{E}_1, \underline{F}_1$ and $\underline{E}_2, \underline{F}_2$ by gluing the outgoing ends of $\underline{E}_1, \underline{F}_1$ to the incoming ends of $\underline{E}_2, \underline{F}_2$ then

$$\text{(34)} \qquad\qquad \Phi(\underline{E}, \underline{F}) = \Phi(\underline{E}_1, \underline{F}_1) \circ \Phi(\underline{E}_2, \underline{F}_2).$$

As in the unquilted case in Remark 4.17, these invariants are independent of the choice of almost complex structure and deformation of the two-form on the total space.

4.4. **Vanishing theorem.** In this section we glue along a seam to obtain a vanishing theorem analogous to [42, Section 2.3] for the invariants associated to standard fibrations associated to a fibered Dehn twist.

Remark 4.24. (Gluing along a seam for quilted Lefschetz-Bott fibrations)

(a) (Glued surface) For $k = 0, 1$ let $\underline{S}_k$ be quilted surfaces with $d_k + 1$ strip-like ends, and z_k a seam point in $\underline{S}_k$. Let $\rho > 0$ be a *gluing parameter*. Let $\underline{S}^\rho$ be the quilted surface with $d_0 + d_1 + 2$ strip-like ends formed by gluing together quilted disks D_0, D_1 around z_0, z_1 using the map $z \mapsto \rho/z$. See Figure 5.
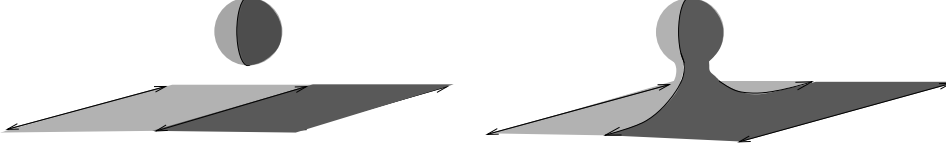

FIGURE 5. Gluing along a seam

(b) (Glued bundles) Let $(\underline{E}_k, \underline{F}_k)$ be Lefschetz-Bott fibrations over $\underline{S}_k$, equipped with a trivialization of $\underline{E}_k, \underline{F}_k$ in a neighborhood of z_k, and a symplecto-morphism of $(\underline{E}_{k,z_k}, \underline{F}_{k,z_k})$ for $k = 0, 1$. The *seam connect sum* $\underline{E}^\rho \to \underline{S}^\rho$ is formed by patching $\underline{E}_0$ and $\underline{E}_1$, and similarly for the boundary and seam conditions $\underline{F}^\rho$.
(c) (Glued complex structures) Suppose that the following are given: a (π_k, j_k)-compatible almost complex structure $\underline{J}_k$ for $\pi_k : \underline{E}_k \to \underline{S}_k$ that is constant in a neighborhood of z_k (with respect to the given trivialization) for $k \in \{0, 1\}$ and such that $\underline{J}_0$ agrees with $\underline{J}_1$ on the glued fiber. One can patch together these almost complex structures to obtain a compatible almost complex structure $\underline{J}$ for $\underline{E} \to S$.

We next investigate the behavior of moduli spaces of sections under gluing. Suppose that $(\underline{E}^\rho, \underline{F}^\rho)$ is obtained from gluing $(\underline{E}_0, \underline{F}_0)$ and $(\underline{E}_1, \underline{F}_1)$ and all three are monotone symplectic Lefschetz-Bott fibrations. Let

$$\mathrm{ev}_k : \ \mathcal{M}(\underline{E}_k, \underline{F}_k) \to \underline{F}_{k,z_k}$$

denote the evaluation maps at the nodal points $z_k, k \in \{0, 1\}$.

Theorem 4.25. (Behavior of moduli spaces under gluing along a seam) *For sufficiently small ρ, there exists a comeager subset*

$$(\mathcal{J}(\underline{E}_0, \underline{F}_0) \times \mathcal{J}(\underline{E}_1, \underline{F}_1))^{\mathrm{reg}} \subset \mathcal{J}(\underline{E}_0, \underline{F}_0)^{\mathrm{reg}} \times \mathcal{J}(\underline{E}_1, \underline{F}_1)^{\mathrm{reg}}$$

such that

(a) *the evaluation map $\mathrm{ev}_0 \times \mathrm{ev}_1$ is transverse to the diagonal;*
(b) *for any pair (u_0, u_1) there exists a* gluing map *on a neighborhood $U(u_0, u_1)$ of (u_0, u_1) given by*

$$\Theta^\rho : \mathcal{M}(\underline{E}_0, \underline{F}_0) \times_{\mathrm{ev}_0, \mathrm{ev}_1} \mathcal{M}(\underline{E}_1, \underline{F}_1) \supseteq U(u_0, u_1) \to \mathcal{M}(\underline{E}^\rho, \underline{F}^\rho);$$

(c) *as (u_0, u_1) varies over points in the zero-dimensional component of the left-hand-side, Θ^ρ is surjective onto the zero-dimensional component of $\mathcal{M}(\underline{E}^\rho, \underline{F}^\rho)$; and*
(d) *for any $u \in \mathcal{M}(\underline{E}_0, \underline{F}_0) \times_{\mathrm{ev}_0, \mathrm{ev}_1} \mathcal{M}(\underline{E}_1, \underline{F}_1)$, the sequence $\Theta^\rho(u)$ Gromov converges to u as $\rho \to 0$, that is, converges up to sphere bubbling, disk bubbling, and bubbling off of Floer trajectories on the strip-like ends.*

See McDuff-Salamon [30, Chapter 10] for the case of gluing at an interior point, and Abouzaid [1] for the details of gluing along a point in the boundary.

Next we give a formula for the dimension for the pseudoholomorphic sections of the standard fibration studied in Propositions 2.13 and 4.20.

Definition 4.26. (Standard fibrations on small balls) Let $C \subset M$ be a coisotropic submanifold of codimension c spherically fibered over B and $E_C \to D$ the Lefschetz-Bott fibration over the disk D of radius 1 with generic fiber $M \times B^-$ and monodromy $\tau_C \times 1$ from Lemma 2.13. Equip E_C with the Lagrangian boundary condition given by the fiber product

$$F_C := P(T) \,|\, \partial D$$

where T is the union of vanishing cycles in the local model $P(V)$ as in (10). As in [42] this boundary value problem fits into a family of problems $E_{C,r}, F_{C,r}$, the standard fibrations of Section 2.4 over a disk D_r of radius r. Each member of the family is formed by patching together $C \times_{SO(c+1)} V$ with $(M \times B) - (i(C) \times p(C))$ with boundary condition

$$F_{C,r} = \bigcup_{z \in \partial D_r} \left(P \times_{SO(c+1)} \sqrt{z} S^c \right)$$

in the local model $P \times_{SO(c+1)} V$.

Lemma 4.27. (Dimension formula for pseudoholomorphic sections of a standard fibration) *Let* C, E_C, F_C *be the standard Lefschetz-Bott fibration associated to* C *in Lemma 2.13, and* c *the codimension of* C. *If* (E_C, F_C) *is monotone then for* r *sufficiently small,* $J \in \mathcal{J}^{\mathrm{reg}}(E_C)$ *and all* $u \in \mathcal{M}(E_{C,r}, F_{C,r})$ *we have*

$$\dim T_u \mathcal{M}(E_{C,r}, F_{C_r}) \geq \dim(C) + (c-1).$$

Proof. We explicitly describe a family of pseudoholomorphic sections as follows. Suppose that the almost complex structure on $E_{C,r}$ is induced from an almost complex structure on B and the standard almost complex structure on V. For each $a \in V, b \in B$ fix a local trivialization of P and define in the corresponding local trivialization $E_{C,r}$

$$w_{r,a,b} : D_r \to E_{C,r}, \quad z \mapsto r^{-1/2} a z + r^{1/2} \overline{a}.$$

The condition for $w_{r,a,b}$ to be a section is by (9)

$$\begin{aligned} z &= w_{r,a,b}(z) \cdot w_{r,a,b}(z) \\ &= (r^{-1/2} a z + r^{1/2} \overline{a}) \cdot (r^{-1/2} a z + r^{1/2} \overline{a}) \\ &= r^{-1} a \cdot a z^2 + 2a \cdot \overline{a} z + r \overline{a} \cdot \overline{a}. \end{aligned}$$

Hence

(35) $$2a \cdot \overline{a} = 1, \quad a \cdot a = \overline{a} \cdot \overline{a} = 0.$$

Given this, the section $w_{r,a,b}$ takes values in $F_{C,r}$. We claim that the Maslov index satisfies the following formula:

(36) $$I(w_{r,a,b}) = c - 1.$$

Indeed, by definition the Maslov index of $w := w_{r,a,b}$ is the index of the pair $(w^*T^{\text{vert}}E_C, (\partial w)^*T^{\text{vert}}F_C)$. This pair fits into the exact sequence

$$0 \to (w^*T^{\text{vert}}E_{C,r}, (\partial w)^*T^{\text{vert}}F_{C,r}) \to (w^*TE_{C,r}, (\partial w)^*TF_{C,r}) \to (TD, T(\partial D)) \to 0.$$

Now $T_{w(z)}E_{C,r} = (T_bB_{C,r})^2 \oplus \mathbb{C}^{c+1}$ is trivial, and the boundary condition has vertical part

$$(37) \qquad T^{\text{vert}}_{w(z)}F_{C,r} \cong \Delta_{T_bB} \oplus T_{w(z)}(\sqrt{z}S^c) \cong \Delta_{T_bB} \oplus \mathbb{R}^{c+1}/\mathbb{R}w(z)$$

where Δ_{T_bB} is the diagonal. The horizontal part of the boundary value problem maps isomorphically onto $(TD, T(\partial D))$ via $D\pi$. Using (37) we have

$$I(w^*TE_{C,r}, (\partial w)^*TF_{C,r}) = I(\mathbb{C}^{c+1}, \sqrt{z}\mathbb{R}^{c+1}).$$

Hence

$$\begin{aligned} I(w^*T^{\text{vert}}E, (\partial w)^*T^{\text{vert}}F) &= I(w^*TE, (\partial w)^*TF) - (TD, T(\partial D)) \\ &= (c+1) - 2 = c - 1 \end{aligned}$$

which proves the claim (36).

The given sections are the sections of lowest index for sufficiently small radius. Indeed, the area $A(w_{r,a,b})$ approaches zero as $r \to 0$, for all a; this fact holds even after the adjustment [42, (1.17)] since the adjustment is by the pullback of the differential of a bounded one-form. Choose r sufficiently small so that $A(w_{r,a,b}) \leq 1/2\lambda$. The area-index monotonicity relation and non-negativity of the curvature in Proposition 4.20 imply that any other section u has positive area. So the index $I(u)$ is at least the index $I(w_{r,a,b})$ of $w_{r,a,b}$. $\qquad\square$

In order to define relative invariants for Lefschetz-Bott fibrations with codimension one vanishing cycles, a stronger monotonicity assumption must be assumed.

Definition 4.28. (Strong monotonicity) In the case that $C \subset M$ is a fibered coisotropic of codimension one we denote by M_C the *cut space* as in Lerman [24]. The cut space M_C is the space obtained by cutting M along C and collapsing the resulting manifold (whose boundary is two copies of C) by the circle action on the boundary. Thus M_C contains two copies $B_\pm$ of B. Denote by $[B_\pm] \in H^2(M_C)$ the dual classes of $B_\pm$ and by $[\omega_C] \in H^2(M_C)$ the symplectic class. A monotone fibered coisotropic C is *strongly monotone* if either $\text{codim}(C) \geq 2$ or both $\text{codim}(C) = 1$ and $c_1(M_C) - [B_+] - [B_-]$ is a positive multiple of $[\omega_C]$.

Proposition 4.29. *Suppose that the vanishing cycles of E are strongly monotone. Then the sections of (E, F) satisfy a monotonicity relation, and counting pseudo-holomorphic sections defines a relative invariant $\Phi(E, F; \Lambda)$ as in (32).*

Proof. The codimension two case is already covered in Proposition 4.10. In the case that the fibered coisotropic is codimension one, there are degree two homology classes in the special fibers that are not equivalent to homology classes in the special fiber. Let $\nu_C : M_C \to E_{s_0} \subset E$ denote the normalization map, mapping the two copies $B_\pm$ of B created by cutting to the singular locus $B \subset E_{s_0}$. Then

$$\nu_C^*c_1(E) = c_1(M_C) - [B_+] - [B_-].$$

Indeed, let z be a local coordinate on the base near a critical value. The one-form $\pi^* \mathrm{d}z$ is non-vanishing everyone except at the singular locus in the fiber where $\pi^* \mathrm{d}z$ has a simple zero. It follows that if the vanishing cycles are strongly monotone, then the monotonicity relation holds on fiber classes. It follows that monotonicity also holds on sections. Counting pseudoholomorphic sections defines an invariant $\Phi(E, F)$, as in the case of higher codimension. $\qquad\square$

Remark 4.30. (Double cover in the codimension one case) In the case that $c = 1$, $r = 1$, the evaluation map on the moduli space $\mathcal{M}(E_{C,r}, F_{C,r})_0$ of zero-index pseudoholomorphic sections of $(E_{C,r}, F_{C,r})$ at $z = 1$ induces a double cover

(38) $$\mathrm{ev}_1 : \mathcal{M}(E_{c,r}, F_{C,r})_0 \to C, \ w_{r,a,b} \mapsto w_{r,a,b}(1) = a + \overline{a}$$

of the fiber of the vanishing cycle C. The relations (35) on $a = (a_1, a_2)$ become

(39) $$a_2 = \pm i a_1, \quad a_1 \overline{a_1} = 1/2.$$

Corollary 4.31. (Vanishing of the relative invariant associated to a standard fibration) *Suppose E is the Lefschetz fibration with the Lagrangian boundary condition F obtained by a seam connect sum from a Lefschetz-Bott fibration $(E_{C,r}, F_{C,r})$ over the disk D_r corresponding to a spherically fibered coisotropic submanifold C, with an arbitrary quilted Lefschetz-Bott fibration $E_0 \to S_0$ with boundary condition F_0. Suppose that all these fibrations with boundary conditions are monotone and equipped with relative spin structures, so that in particular the relative invariant $\Phi(E, F)$ is defined. Then $\Phi(E, F) = 0$. More precisely, there exists a null-homotopy of the chain-level operator $C\Phi(E, F)$ that, if r is sufficiently small, has positive q-exponents.*

Proof. Suppose that E, F are as in the statement of the Corollary. Consider the family of surfaces obtained by stretching the neck so that a standard fibration $(E_{C,r}, F_{C,r})$ bubbles off. Since the fibration is trivial outside of a small neighborhood of the singular fiber, the two-forms $\omega_{E_{C,r}}$ in this deformation are diffeomorphic. The chain level invariants $C\Phi(E_{C,r}, F_{C,r})$ are chain-homotopic by a chain-homotopy corresponding to isolated points in the parametrized moduli space for the deformation, q-weighted by symplectic area. Since the curvature is non-negative and zero-area sections are of even index, the q-exponents in the homotopy operator are positive. The gluing theorem 4.25 gives a bijection between pseudoholomorphic sections of E_ρ, F_ρ, for small ρ, and (u_0, u_1) in the zero-dimensional component $(\mathrm{ev}_0, \mathrm{ev}_1)^{-1}(\Delta)$, where Δ is the diagonal in the gluing fiber $F_{0,z_0} \cong F_{C,z_1}$. Equality of indices gives

$$\dim T_{u_0} \mathcal{M}(E_0, F_0) + \dim T_{u_1} \mathcal{M}(E_C, F_C) = \dim(C)$$

for $k \in \{0, 1\}$. In the codimension two case Corollary 4.31 now follows from Lemma 4.27 which implies that the moduli spaces for small r and large ρ are empty. The chain-level invariants are independent of the choice of r up to chain-homotopy.

The codimension one case depends on the description of the double cover in 4.30. Indeed, in this case the limiting moduli space is not empty, but has two components corresponding to the choices of sign in 4.30. Each component is isomorphic to the Lagrangian. Furthermore, each has the induced orientation given by deforming the boundary value problem to a trivial one and using the orientation from the

Lagrangian as in Remark 4.13. The two sections $u_\pm$ corresponding to a given (a_1, a_2) in (39) are related by the involution

$$\iota : \mathbb{C}^2 \to \mathbb{C}^2, \quad (z_1, z_2) \mapsto (z_1, -z_2).$$

It follows that any deformation of $u_+^* F$ to a trivial boundary condition induces a deformation of $u_-^* F$ and vice versa, giving a commutative diagram

$$
\begin{array}{ccc}
\det(D_{u_+^* E, u_+^* F}) & \longrightarrow & \det((u_+^* F)_1) \\
\downarrow & & \downarrow \\
\det(D_{u_-^* E, u_-^* F}) & \longrightarrow & \det((u_-^* F)_1).
\end{array}
$$

The evaluation maps for $u_\pm$

$$\ker D_{u_+^* E, u_+^* F} \cong u_+^* F_1, \quad \ker D_{u_-^* E, u_-^* F} \cong u_-^* F_1$$

given by linearizing (35) are by definition orientation preserving. Since the right-hand-arrow is orientation-reversing (exactly one factor of F is reversed) the left-hand-arrow is also orientation-reversing. Hence these contributions cancel in the weighted disk count (31). $\qquad\square$

Remark 4.32. (Independence of choices) Let $S_0, E_0, F_0, E_C, F_C, D_r$ be as above. Define an invariant associated to the nodal surface S_r by identifying a point on the seam S_0 and disk D. Define a nodal Lefschetz fibration E_r by identifying a fiber of E_0 with a fiber of E_C with boundary condition $F_r \subset E_r$. By counting pseudoholomorphic sections with matching condition at the node one obtains an invariant $\Phi(E_r, F_r)$ for the nodal fibration, equal to the invariant for the glued fibration $\Phi(E, F)$. In particular, vanishing of $\Phi(E_r, F_r)$ for r small implies vanishing for any r.

4.5. **Horizontal invariants.** As in Seidel [42], the computation of the relative invariants in the special cases needed for the exact triangle uses only horizontal sections, defined as follows.

Definition 4.33. Let $\underline{S}$ be a quilted surface with strip-like ends, $\underline{E}$ a quilted Lefschetz-Bott fibration, and $\underline{F}$ a collection of Lagrangian seam/boundary conditions.

    (a) (Horizontal sections) A section $u : \underline{S} \to \underline{E}$ (that is, a collection of sections $u_k : S_k \to E_k$, $k = 1, \ldots, m$) is *horizontal* if

$$\operatorname{Im} Du_k(s) = TE_{k, u_k(s)}^h$$

    for all $s \in S_k, k = 1, \ldots, m$. Let $\mathcal{M}^h(\underline{E}, \underline{F})$ denote the space of horizontal sections.

    (b) (Horizontal almost complex structures) A collection of compatible almost complex structures $\underline{J} \in \mathcal{J}(\underline{E})$ is *horizontal* if

$$J_k(e)T_e^h E_k = T_e^h E_k, \quad \forall e \in E_k - E_k^{\mathrm{crit}}, k = 1, \ldots, m$$

or equivalently if $\omega_{E_k}(J\cdot,\cdot)$ is symmetric for $k = 1,\ldots,m$. Let $\mathcal{J}^h(\underline{E})$ denote the set of horizontal compatible almost complex structures. If $J \in \mathcal{J}^h(\underline{E})$, then any horizontal section is J-holomorphic, that is,

$$(J \in \mathcal{J}^h(\underline{E})) \implies (\mathcal{M}^h(\underline{E},\underline{F}) \subset \mathcal{M}(\underline{E},\underline{F})).$$

(c) (Clean moduli spaces) $\mathcal{M}^h(\underline{E},\underline{F})$ is *clean* if $\mathcal{M}^h(\underline{E},\underline{F})$ is a smooth submanifold of the Banach manifold of sections of a sufficiently differentiable Sobolev class of $\underline{E}$ with tangent space

$$T_u\mathcal{M}^h(\underline{E},\underline{F}) = \{\xi \in \Omega^0(u^*T^{\mathrm{vert}}\underline{E},(u|\partial\underline{S})^*T^{\mathrm{vert}}\underline{F}) \mid \nabla_u\xi = 0\}$$

the set of horizontal sections of $u^*T^{\mathrm{vert}}\underline{E}$ with boundary/seams in $(u|\partial\underline{S})^*T^{\mathrm{vert}}\underline{F}$.

Remark 4.34. (Horizontal and vertical energy) The energy of a section of a Lefschetz-Bott fibration may be broken up into horizontal and vertical parts. We suppose that we choose ω_S so that $\omega_E + \pi^*\omega_S$ is symplectic. This two-form together with the almost complex structure defines a metric on the total space E. Fix a metric on S, giving rise to an area form $\mathrm{d}\,\mathrm{Vol}_S$, and define

$$\mathcal{E}(u) = \int_S \|\mathrm{d}u\|^2 \mathrm{d}\,\mathrm{Vol}_S.$$

The energy $\mathcal{E}(u)$ splits into vertical and horizontal contributions

$$\mathcal{E}(u) = \mathcal{E}^v(u) + \mathcal{E}^h(u), \quad \mathcal{E}^{v,h}(u) = \int_S \|\mathrm{d}^{v,h}u\|^2 \mathrm{d}\,\mathrm{Vol}_S.$$

As in Seidel [42, Equation (2.10)] one has an energy-area relation

$$\mathcal{E}(u) = \int_S u^*(\omega_E + \pi^*\omega_S) + (1/2)\int_S \|\overline{\partial}_J u\|^2 \mathrm{d}\,\mathrm{Vol}_S.$$

This implies that for pseudoholomorphic sections one has an energy-area relation

(40) $$\mathcal{E}^v(u) + \int_S f(u)\omega_S = \int_S u^*\omega_E.$$

A similar relation holds in the quilted case.

Proposition 4.35. (Criterion for horizontal moduli spaces to be a component) *Suppose that $(\underline{E},\underline{F})$ is a quilted symplectic Lefschetz-Bott fibration with Lagrangian boundary condition. If $\underline{E}$ is non-negatively curved and equipped with a horizontal almost complex structure such that $\mathcal{M}^h(\underline{E},\underline{F})$ is clean, of dimension $\mathrm{Ind}(D_u)$, and consists of sections with vanishing ω_E-integral,*

$$\forall u \in \mathcal{M}^h(\underline{E},\underline{F}), \quad \int_S u^*\omega_E = 0$$

then $\mathcal{M}^h(\underline{E},\underline{F})$ is a path component of $\mathcal{M}(\underline{E},\underline{F})$ and consists of regular sections.

Proof. As in [42, Lemma 2.11], suppose that u is a horizontal section with $\int_S u^*\omega_E = 0$. Any other pseudoholomorphic section $\tilde{u}$ homotopic to u satisfies by (40) the equation

$$\mathcal{E}^v(\tilde{u}) + \int_S f(\tilde{u})\omega_S = 0.$$

By the non-negative curvature assumption $\tilde{u}$ must satisfy

$$\mathcal{E}^v(\tilde{u}) = 0, \quad \operatorname{im}(\mathrm{d}\tilde{u}) \subset T^h E$$

as claimed. The regularity of $\mathcal{M}^h(\underline{E}, \underline{F})$ follows as in Seidel [42, Lemma 2.12]: Taking the second derivative of the index-area relation (40) gives

$$\|\nabla_u \xi\|^2 + \int_S \operatorname{Hess}(f)(u)\omega_S = \|D_u \xi\|^2$$

where Hess is the Hessian of f. Non-negativity of the curvature implies that the Hessian is positive, and so $\ker(D_u) \subset \ker(\nabla_u)$. Then $\dim \mathcal{M}^h(\underline{E}, \underline{F}) = \operatorname{Ind}(D_u)$ implies $\operatorname{coker}(D_u) = 0$. $\qquad\square$

We give a criterion for the zero-dimensional component $\mathcal{M}(\underline{E}, \underline{F})_0$ to consist entirely of horizontal sections in the monotone case; the exact case was discussed in Seidel [42]. Let $\underline{E} \to \underline{S}$ be a monotone Lefschetz-Bott fibration with Lagrangian boundary/seam conditions $\underline{F}$. For each collection of intersection points $((x_e)_{e \in \mathcal{E}})$ define $c((x_e)_{e \in \mathcal{E}})$ by

$$\lambda I(u) = 2(A(u) - c((x_e)_{e \in \mathcal{E}}))$$

for any collection of sections u with limits $(x_e)_{e \in \mathcal{E}}$.

Proposition 4.36. (Criterion for the existence of only horizontal sections) *Suppose that $\underline{E}$ is a quilted Lefschetz-Bott fibration with non-negative curvature and is equipped with a horizontal almost complex structure. If $c((x_e)_{e \in \mathcal{E}}) = 0$ then $\mathcal{M}(\underline{E}, \underline{F}; (x_e)_{e \in \mathcal{E}})_0$ consists of horizontal sections.*

Proof. If u is a pseudoholomorphic section with index 0, then the monotonicity relation (29) implies that u has non-negative symplectic area equal to $c((x_e)_{e \in \mathcal{E}})$. If this constant vanishes, then all such sections must have vanishing symplectic area. By the non-negativity of the curvature and (40), any such section has vanishing vertical energy and so is horizontal. $\qquad\square$

5. Floer versions of the exact triangle

The proofs of the exact triangles described in the introduction follow the lines of the proof of Floer's exact triangle [6], in Seidel [42] and Perutz [36]. Namely, one first constructs a short sequence of cochain groups that is exact *up to leading order*, and then uses a spectral sequence argument to deduce the existence of a long exact sequence of cohomology groups. In this section we also describe various extensions, to the case of minimal Maslov number two and the case of periodic Floer cohomology.

5.1. **Fibered Picard-Lefschetz formula.** In this section we prove the exact triangle on the level of vector spaces; this is essentially equivalent to the fibered Picard-Lefschetz formula in Theorem 1.

Definition 5.1. (Angle functions) Recall from Section 2.2 that a Dehn twist in a local model $T^\vee S^c$ is defined by rotating a vector in $T^\vee S^c$ with norm t by an *angle function*

$$\theta \in C^\infty([0, \infty)), \quad \theta(0) = \pi, \quad \theta(t) = 0, t \gg 0.$$

The angle function $\theta(t)$ is related to the choice of $\zeta(t)$ of (8) by $\theta(t) = \zeta'(t)$. For the rest of the paper we assume that the function $\theta(t)$ is decreasing with t. We wish to use angle functions that go to 0 sufficiently quickly. In particular, given an angle function θ we consider the family of angle functions defined by

$$(41) \qquad\qquad \theta^\delta(t) := \theta(\delta t).$$

Lemma 5.2. (Intersection points for fibered Dehn twists) *Let $C \subset M$ be a compact spherically fibered coisotropic in a symplectic manifold M over a symplectic manifold B, and $L^0, L^1 \subset M$ Lagrangian submanifolds. Suppose that the intersection*

$$(L^0 \times C) \pitchfork (C^t \times L^1) \subset M \times B \times M$$

is transverse while the intersection

$$L^0 \cap L^1 \cap C = \emptyset$$

is empty. For $\delta > 0$ let $\tau_{C,\delta} \in \mathrm{Diff}(M, \omega)$ be a fibered Dehn twist using a given local model and the angle function $\theta^\delta(t)$. There exists a constant δ_0 such that if $\delta > \delta_0$ then there exists a bijection

$$(42) \qquad (i_1 \cup i_2) : (L^0 \cap L^1) \cup \big((L^0 \times C) \cap (C^t \times L^1)\big) \to L^0 \cap \tau_{C,\delta}^{-1} L^1.$$

Proof. The intersection points of the untwisted Lagrangians naturally include in the intersection points of the twisted Lagrangians for δ sufficiently large: The transversality assumptions imply that the intersection $L^0 \cap L^1 \cap C$ is empty. By taking δ_0 sufficiently large we may suppose that $\tau_{C,\delta}$ is supported in a neighborhood U_C disjoint from $L^0 \cap L^1$. Thus there is an inclusion

$$i_1 : L^0 \cap L^1 \to L^0 \cap \tau_{C,\delta}^{-1} L^1.$$

We wish to identify the remaining intersection points with $(L^0 \times C) \cap (C^t \times L^1)$.

We first recall the unfibered case in Seidel [42, Lemma 3.2]. Suppose that the base B is a point. Choose a local model for the Lagrangian C given by a symplectomorphism of a neighborhood of the zero section in $T^\vee S^c$ with a neighborhood of C in M. The local model $\phi : T^\vee S^c \supset U \to M$ may be chosen so that the Lagrangians L^0, L^1 are fibers of $\pi : T^\vee S^c \to S^c$:

$$L^k \cap \phi(U) = \phi(U \cap \pi^{-1}(v_k)), \quad v_k \in S^c.$$

What this amounts to is choosing the initial diffeomorphism with this property and then choosing the Moser isotopy to preserve the Lagrangians. The Dehn twist τ_C^{-1} acts at $[v] \in P(T^\vee S^c)$ by normalized geodesic flow by time $\theta^\delta(|v|)$. There exists a unique $v \in L^1$ of norm less than π such that its time $\pi - |v|$ normalized geodesic flow lies in L^0. The unique point $w \in \mathbb{R}_{>0} v$ with $\theta^\delta(|w|) = \pi - |v|$ gives the desired intersection point.

We reduce to the case of trivial base by the use of suitable local models as follows. Let

$$l_0 \in L^0 \cap C, l_1 \in L^1 \cap C, \quad \pi(l_0) = \pi(L_1) =: b$$

be points with the same projection $b \in B$. Consider the action of scalar multiplication on the vector bundle $P(\pi_{T^\vee S^c}) : P(T^\vee S^c) \to P(S^c)$. This action induces an action of sufficiently small scalars on M via the coisotropic embedding. By (41)

$\tau_{C,\delta} = \delta\tau_C\delta^{-1}$ wherever the right-hand-side is defined. For $k \in \{0,1\}$, the submanifolds $(\delta L^k \cap U_C)_{\delta>0}$ fit together with $P(\pi_{T^\vee S^c})^{-1}(L^k \cap C)$ in the limit $\delta \to \infty$ to a smooth family at $\delta = \infty$. Indeed, if L^k is given locally by $\{f(z,x,y) = 0\}$ in local coordinates z on B and cotangent coordinates (x,y) on $T^\vee S^c$ then

$$(43) \qquad\qquad \delta L^k = \{f(z,x,\delta^{-1}y) = 0\}.$$

By assumption L^k is transversal to the zero section in $T^\vee S^c$. This implies that the equation (43) cuts out a smooth family at $\delta = \infty$. By the case B trivial, the intersection at $\delta = \infty$ is given by

$$P(\pi_{T^\vee S^c})^{-1}(L^0 \cap C) \cap \tau_C^{-1}P(\pi_{T^\vee S^c})^{-1}(L^1 \cap C)$$

and is transverse. By the implicit function theorem, the set of intersection points $L^0 \cap \tau_{C,\delta}^{-1}L^1$ forms a smooth manifold parametrized by $\delta \gg 0$ and intersection points $(L^0 \times C) \cap (C^t \times L^1)$ as desired.

Alternatively, one can argue as follows via the limit of the twisted Lagrangian. After restricting to connected component in an open neighborhood of $p^{-1}(b)$, the projection $L^1 \cap C \to B$ is an embedding. For any point $l \in L^1$ with $\tau_C(l) \neq l$, the images of the derivatives of $\tau_{C,\delta}^{-1}$ converge to the tangent space of $p^{-1}(C \cap L^1)$: If the projection of $\tau_C^{-1}(l)$ to C is c in the local model then

$$\mathrm{Im}(D\tau_{C,\delta}^{-1}(\delta^{-1}l)) = \mathrm{Im}(\delta^{-1}D\tau_C^{-1}(l)\delta) \to D_c p^{-1}(T_{p(l)}(C \cap L^1)), \quad \delta \to \infty$$

since $\delta^{-1}D\tau_C^{-1}(l)\delta$ converges to the composition of $D\tau_C^{-1}(l)$ with projection to the tangent space to C. It follows that the sequence of submanifolds $\tau_{C,\delta}^{-1}(L^1) - L^1$ converges, as an embedded submanifold, to $p^{-1}(p(L^1 \cap C)) - L^1$. Again by the implicit function theorem, the intersections $L^0 \cap \tau_{C,\delta}^{-1}(L^1)$ correspond to intersection points of $L^0 \cap p^{-1}(p(L^1 \cap C)) \cong (L^0 \times C) \cap (C^t \times L^1)$. $\qquad\square$

5.2. **Lagrangian Floer version.** We now prove the exact triangle Theorem 1.3. Since τ_C is a symplectomorphism, it suffices to prove the theorem with L^1 replaced by $\tau_C^{-1}L^1$. That is, we construct a long exact sequence

$$\ldots HF(L^0, C^t, C, \tau_C^{-1}L^1) \to HF(L^0, \tau_C^{-1}L^1) \to HF(L^0, L^1)\ldots.$$

More generally in the quilted case, it suffices to prove the theorem with $\underline{L}^1$ replaced by $\underline{L}^1 \# \mathrm{graph}(\tau_C)$.

5.2.1. *Definition of the maps.* Let M be a monotone symplectic background and $C \subset M$ a spherically fibered coisotropic submanifold of codimension $c \geq 2$. Let $L^0, L^1 \subset M$ be admissible Lagrangian branes, and suppose that C is equipped with an admissible brane structure as a Lagrangian submanifold of $M^- \times B$. These conditions imply that all Lefschetz-Bott fibrations discussed below are monotone as in Lemma 4.10. We may assume, after Hamiltonian perturbation, that C, L^0, L^1 all intersect transversally.

Definition 5.3. (a) (Chaps map) The first map in the exact sequence is defined as the relative invariant associated to a "quilted pair of pants", or more accurately, "quilted chaps" in American dialect. Let $\underline{S}_1$ denote the quilted

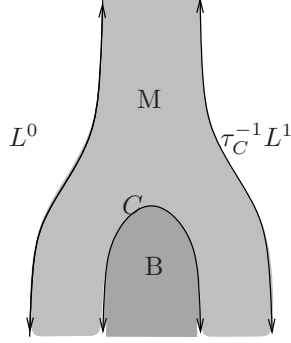surface shown in Figure 6: Let (S_B, S_M) denote the patches of $\underline{S}_1$, and

FIGURE 6. Quilted surface $\underline{S}_1$ defining $HF(L^0, C^t, C, \tau_C^{-1} L^1; \Lambda) \to HF(L^0, \tau_C^{-1} L^1; \Lambda)$

$\underline{E} = (S_M \times M, S_B \times B)$, where B is the base of the fibration $p : C \to B$. We identify C with its image in $M \times B$. Let $\underline{F}$ denote the Lagrangian seam/boundary condition for $\underline{E}$ given by L^0, L^1, C and consider the relative invariant

(44) $\Phi_1 : HF(L^0, C, C^t, \tau_C^{-1} L^1; \Lambda)[\dim(B)] \to HF(L^0, \tau_C^{-1} L^1; \Lambda).$

This invariant was defined in Definition 4.23 by counting points in the zero-dimensional component of the moduli space $\mathcal{M}_1$ of pseudoholomorphic quilts on $\underline{S}_1$.

(b) (Lefschetz-Bott map) The second map in the exact sequence is a relative invariant associated to a Lefschetz-Bott fibration with monodromy given by the Dehn twist. Namely let $E_{C,r} \to D_r$ denote a standard Lefschetz-Bott fibration with monodromy τ_C from Lemma 2.13 and Definition 4.26. Gluing in $E_{C,r}$ with the trivial fibration over a strip (using the identity as transition map to the left of the disk, and τ_C as transition map to the right) as in Seidel [42, p. 7] gives a Lefschetz-Bott fibration (E_2, F_2) over the infinite strip shown in Figure 7. Let

(45) $\Phi_2 : HF(L^0, \tau_C^{-1} L^1; \Lambda) \to HF(L^0, L^1; \Lambda)$

denote the associated relative invariant. Relative invariants were defined in (32) by counting points in the zero-dimensional component of the moduli space $\mathcal{M}_2$ of pseudoholomorphic sections of $E_2 \to S_2$ with boundary in F_2. (It follows from Theorem 5.5 below that Φ_2 has degree zero.)

The first step in the proof of the exact sequence is to show that the composition $\Phi = \Phi_2 \circ \Phi_1$ (the relative invariant associated to picture on the left in Figure 8) vanishes:

Lemma 5.4. (Exactness at the middle term) *The composition $\Phi = \Phi_2 \circ \Phi_1$ (the relative invariant associated to picture on the left in Figure 8) vanishes; more precisely there exists a null-homotopy of the chain level maps $C\Phi_2 \circ C\Phi_1$ whose terms have positive q-exponent for r sufficiently small.*
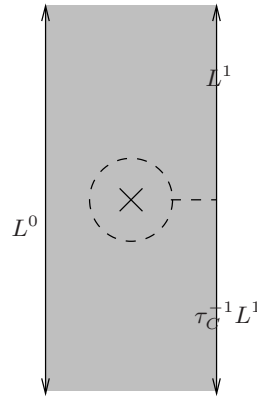
FIGURE 7. Lefschetz-Bott fibration $E_2 \to S_2$ defining the map $HF(L^0, \tau_C^{-1} L^1; \Lambda) \to HF(L^0, L^1; \Lambda)$

Proof. The composition of the two relative invariants is the relative invariant associated to a Lefschetz-Bott fibration over the glued surface $\underline{S} = \underline{S}_1 \# \underline{S}_2$ by (34). Consider the deformation $\underline{S}_t$ of $\underline{S}$ obtained by moving the critical value of the Lefschetz-Bott fibration towards the boundary marked C and pinching off a disk in $M \times B$ with boundary values in C. This process is shown in the right-most two pictures in Figure 8. The bundles $\underline{E}$ and Lagrangian boundary/seam conditions $\underline{F}$ naturally extend to families $\underline{E}_t, \underline{F}_t$ that are obtained from gluing for $t \gg 0$. It follows from Corollary 4.31 that the relative invariant $C\Phi$ is null-homotopic and, for r sufficiently small, has positive q-exponents. $\qquad \square$
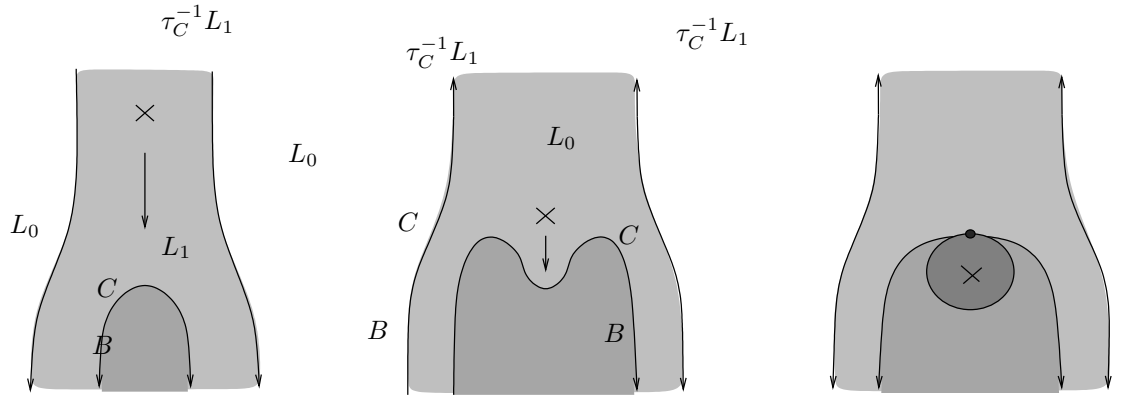
FIGURE 8. Pinching off a disk at the seam

5.2.2. *Exactness to leading order.* The proof that the maps Φ_1, Φ_2 of (44), (45) fit into a long exact sequence follows a standard argument, familiar from Floer's exact triangle [6]. In this argument one first proves that the "leading order terms" in the cochain-level map define a short exact sequence and then applies a spectral sequence

to deduce the triangle. Recall that $L^0 \cap \tau_C^{-1} L^1$ is the disjoint union of the images of i_1 and i_2 of the map in Proposition 5.2.

Theorem 5.5. (Exactness of the short sequence to leading order) *Let* C, L^0, L^1 *be as in Theorem 1.3. There exists* $\epsilon > 0$ *such that for any fibered Dehn twist* τ_C *defined using* (41) *for* λ *sufficiently large, the following properties hold:*

(a) *(Small triangles as leading order contributions to* $C\Phi_1$*) If* $x \in ((L^0 \times C) \cap (C^t \times \tau_C^{-1} L^1))$ *then* $C\Phi_1(\langle x \rangle)$ *is equal to* $q^\nu \langle i_1(x) \rangle$ *for some* $\nu < \epsilon/2$*, plus terms of the form* $q^\mu \langle z \rangle$ *with* $\mu > \epsilon$ *and* $z \in L^0 \cap \tau_C^{-1} L^1$*.*

(b) *(Horizontal sections as leading order contributions to* $C\Phi_2$*)*

    (i) *For any* $x \in L^0 \cap L^1$*, if* $y = i_2(x)$ *then* $C\Phi_2(\langle y \rangle)$ *is equal to* $\langle x \rangle$ *plus terms of the form* $q^\nu \langle z \rangle$ *with* $z \in \mathcal{I}(L^0, L^1)$ *and* $\nu > \epsilon$*.*

    (ii) *If* $y \neq i_2(x)$ *for any* x*, then* $C\Phi_2(\langle y \rangle)$ *is a sum of terms of the form* $q^\nu \langle z \rangle$ *for* $z \in \mathcal{I}(L^0, L^1)$ *and* $\nu > \epsilon$*.*

Furthermore, $C\Phi_2$ *has zero degree.*

Proof. (a) We aim to reduce to the exact, unfibered case considered by Seidel [42].

Step 1: The antipodal case. The simplest case is that in which the points on the spherical fiber are antipodes. That is, let $x = (m, b, m')$ be as in the statement of the Theorem, so that

(46) $$m \in L^0 \cap C, \ m' \in L^1 \cap C, \ \pi(m) = \pi(m') \in B.$$

Suppose that m, m' are antipodes:

$$b = \pi(m) = \pi(m'), \quad L^0 \cap \tau_C^{-1} L^1 = \{m\}.$$

Let

$$u : \underline{S}_1 \to \underline{M}, \quad u|S_M \equiv m, \quad u|S_B \equiv b$$

denote the constant section with value m on the part mapping to M and value b on the part mapping to B. This section satisfies the boundary conditions $L^0, \tau_C^{-1} L^1, C$, by (46).

We claim that these constant sections give the only contributions to the relative invariant in the antipodal case. Let $J \in \mathcal{J}^h(\underline{E})$ be any horizontal almost complex structure, for which u is a pseudoholomorphic section with zero area:

$$A(u) = \int_{\underline{S}} u^* \underline{\omega} = A(u|S_M) + A(u|S_B) = 0.$$

The map u is J-regular. Indeed, by non-negativity of the curvature and Proposition 4.35, it suffices to show that the index of the linearized operator $D_{u,J}$ for u the constant section is zero. Since the boundary conditions are constant, we may decompose the boundary conditions into vertical and horizontal parts,

$$T_m M \cong T_b B \oplus T_v T^\vee S^c, \quad T_m C \cong T_b B \oplus T_v S^c$$

for some vector $v \in S^c$ representing m in the trivialization of the fiber at b. The linearized operator D_u breaks into the sum of linearized operators D_u^h, D_u^v for the horizontal and vertical pieces. The linearized operator D_u^h for the horizontal piece $T_b B$ may be identified with a linearized operator on a strip with constant, transverse

boundary conditions $T_b(L_0 \circ C)$ and $T_b(L_1 \circ C)$. The kernel and cokernel of D_u^h are trivial, hence the horizontal piece D_u^h has index zero. The linearized operator for the vertical piece D_u^v is equivalent to that for the unfibered case covered in [42, Proof of Lemma 3.3]. Since the kernel and cokernel in this case is also trivial, the cokernel of the linearized operator for u is trivial. Hence J is regular for horizontal u. This implies that we may use J to compute the coefficient of $\langle i_1(x) \rangle$ in $C\Phi_1(\langle x \rangle)$, and the fact that the required sections are constant in this case proves the claim.

Step 2: Deformation to the antipodal case. We reduce to the antipodal case of the previous paragraphs by a deformation argument. Suppose that (m, b, m') gives a generalized intersection point of (L^0, C^t, C, L^1). Choose a family of identifications $C_b \cong S^c$ depending on $t \in [0, 1]$ such that for $t = 1$ the points m, m' are antipodes in C_b. We extend the pull-back metrics on C_b to a family of fiber-wise metrics on C. The family of metrics defines a reduction of the structure group depending on t, that is, a family of principal $SO(c+1)$ bundles P_t together with a family of diffeomorphisms

$$P_t \times_{SO(c+1)} S^c \to C.$$

Let τ_C^t denote the resulting family of fibered Dehn twists, and consider the family of boundary conditions given by $\tau_C^t L^0, L^1$. The intersection points $\mathcal{I}(\tau_C^t L^0, C^t, C, L^1)$ fit into smooth families depending on t as in the proof of Proposition 5.2. Consider the parametrized moduli space

$$(47) \qquad \widetilde{\mathcal{M}}(i_1(x), x) = \bigcup_{\rho \in [0,1]} \{\rho\} \times \mathcal{M}^\rho(i_1(x), x)$$

of pseudoholomorphic curves for this deformation. Standard arguments show that there is a parametrized regular family of almost complex structures for the deformation, in the sense that the moduli space (47) is a smooth, finite dimensional manifold with boundary. The moduli space is also compact, as long as δ is sufficiently large. In this case bubbling off trajectories in the deformation (47) is impossible by Remark 5.6 below. On the other hand, bubbling off holomorphic disks and spheres is impossible because of the monotonicity conditions. It follows from compactness that the count of holomorphic quilts is invariant under the deformation (47).

We show that the sections contributing to the coefficient of $\langle i_1(x) \rangle$ in $C\Phi_1(\langle x \rangle)$ are of small area. Suppose $u : \underline{S}_1 \to \underline{M}$ is a quilt contained in a neighborhood of C_b with Lagrangian boundary conditions $L^0, \tau_C^{-1} L^1, C$ and limits $x, i_1(x)$. Near C_b all Lagrangians are exact and path connected. The area computation reduces to the computation of action differences in [42]. In particular, for these maps, for δ sufficiently large, all such u of index 0 (parametrized index 1) connecting $i_1(x)$ with x have $\mathcal{E}(u)$ at most $\epsilon/2$:

$$u \in \mathcal{M}(x, i_1(x)) \implies \mathcal{E}(u) \le \epsilon/2.$$

This argument shows that the area of small index zero quilts is at most $\epsilon/2$ for δ sufficiently large. By monotonicity, any holomorphic quilt of index zero with boundary/seam conditions $L^0, \tau_C^{-1} L^1, C$ has the same area as one of the index zero quilts above.

It follows in particular that sphere and disk bubbling does not occur in these moduli spaces. So the component of the moduli space $\widetilde{\mathcal{M}}_1(i_1(x), x)$ of formal dimension one is compact. Counting boundary components gives

$$(48) \qquad \sum_{u \in \mathcal{M}^{\rho=0}(i_1(x), x)_0} o(u) = \sum_{u \in \mathcal{M}^{\rho=1}(i_1(x), x)_0} o(u)$$

where $o(u) = \pm 1$ are the orientations. The second sum in (48) is equal to 1 if the almost complex structure is horizontal. For the unique contribution then comes from the horizontal section with value $i_1(x) = (m, b, m')$. This completes the deformation argument.

Step 3: The higher order terms. We next show that the pseudoholomorphic sections with limits other than those corresponding to i_1, i_2 have higher energy, using the mean value inequality. Suppose that $u = (u_0, u_1)$ is a quilt from $\underline{S}_1$ connecting $x = (m, b, m')$ with y with energy at most ϵ with u_1 resp. u_0 mapping to M resp. B. For any neighborhood U of C_b, there exists $\epsilon > 0$ sufficiently small so that the image of u_1 is contained in U:

$$\mathcal{E}(u) < \epsilon \implies u_1(S_M) \subset U.$$

Indeed let (s_0, t_0) denote the coordinate of the top-most point in the seam. For $s > s_0 + 1$, integrating the mean value inequality (30) as in Lemma 4.15 over a path $\gamma_s(t) = (s, t)$ shows that the distance between L^0 and C and between $\tau_C^{-1} L^1$ and C are at most $c\epsilon$ for some constant c. In fact these estimates are independent of the choice of τ_C. Indeed, the image $\tau_C^{-1}(L^1)$ is independent of τ_C in a neighborhood of $L^0 \cap L^1$. On the other hand, $\tau_C^{-1}(L^1)$ converges to $\pi^{-1}\pi(L^1 \cap B)$ as $\zeta(0) \to 0$. As in Lemma 4.4, let V_ϵ be the set of points in X within distance $c\epsilon$ of both L^0 and $\tau_C^{-1} L^1$. Since $L^0, \tau_C^{-1} L^1$ are compact, for ϵ sufficiently small, V_ϵ is a collection of disjoint open neighborhoods of the points in $L^0 \cap \tau_C^{-1} L^1$:

$$V_\epsilon = \bigcup_{z \in L^0 \cap \tau_C^{-1} L^1} V_\epsilon(z).$$

It follows that if u has sufficiently small energy then every point with $s > s_0 + 1$ maps to V_ϵ, hence $V_\epsilon(x)$. Similarly, for $s < s_0 - 1$, one obtains by "folding" a strip with values in $M \times B \times M$ with boundary values $L^0 \times C$ and $C^t \times \tau_C^{-1} L^1$. Integrating the mean value inequality (30) over paths to the boundary $\gamma_s(\tau) = (s, t + \tau)$ or $\gamma_s(t) = (s, t - \tau)$ shows that

$$d(\pi_1 u(s, t), L^0) < c\epsilon, \quad d(\pi_2 u(s, t), C) < c\epsilon, \quad d(\pi_3 u(s, t), \tau_C^{-1} L^1) < c\epsilon.$$

In particular note that the intersection $(L^0 \circ C) \cap (\tau_C^{-1} L^1 \circ C)$ is transverse and independent of τ_C. Hence the projections $\pi_1(u(s, t))$ and $\pi_3(u(s, t))$ are contained a small neighborhood W of $\pi^{-1}((C \circ L^0) \cap (C \circ \tau_C^{-1} L^1))$ in M. It follows that u takes values in W for all $s < s_0 - 1$. Applying Lemma 4.15 again, this time for a path of the form $\gamma(t) = [s_0 + 2t, k]$ for $k \in \{0, 1\}$, shows the same holds for points in the boundary of the intermediate region. Now $i_1(x)$ is the unique intersection point of

$L^0 \cap \tau_C^{-1} L^1$ connected to $L^0 \cap C_b$ by path from m in $L^0 \cap W$, and a path from m' in $\tau_C^{-1} L^1 \cap W$, as in the proof of Proposition 5.2. Thus $y = i_1(x)$.

(b) The second part of the Lemma is somewhat easier, since the leading order terms have order exactly zero arising from the horizontal sections.

Step 1: We show that the degree zero terms arise from horizontal sections. Let u be the horizontal section of E_2 on the infinite strip with value x. Then $\mathrm{Ind}(D_{u,J}) = 0$, since the boundary conditions are constant. Hence u is regular for horizontal J, and the count for $y = i_1(x)$ follows by Proposition 4.36. The map Φ_2 has degree 0, since the horizontal sections have zero index. Because of the non-negative curvature of the standard Lefschetz-Bott fibration in 4.20, any non-horizontal section has positive area. Thus the q-exponent-zero terms arise only from horizontal sections.

Step 2: The non-horizontal sections satisfy a uniform energy gap condition as in the Lemma. For an almost complex structure pulled back from one for (L^0, C^t, C, L^1) under $\mathrm{Id}_M \times \mathrm{Id}_B \times \tau_C^{-1}$; this is an immediate consequence of Gromov compactness and preservation of area and index under the pull-back. The following alternative argument using the mean value inequality holds for any almost complex structure used to define the relative invariant: Suppose u is a section of energy less than ϵ. By Lemma 4.15, for ϵ small u takes values in the set U of points in M within distance $c\epsilon$ of both L^0 and L^1. By Lemma 4.4, U_ϵ is a collection of disjoint open neighborhoods of the points in $L^0 \cap L^1$:

$$U_\epsilon = \bigcup_{z \in L^0 \cap L^1} U_\epsilon(z).$$

Since $u(s,t)$ lies in $U_\epsilon(x)$ for s sufficiently large, this implies that

$$u(s,t) \in U_\epsilon(x), \quad s \geq 1.$$

For the points in $[-1,1] \times \{0,1\}$ integration applied to the path from $(s,0)$ to $(1,0)$ to $(-1,1)$ shows that (again for ϵ sufficiently small)

$$u(s,0) \in U_\epsilon(x), \quad s \in [-1,1].$$

Then another application of integration shows that

$$u(s,t) \in U_\epsilon(x), \quad s \leq -1$$

as well. Thus u has the same limit x at $s = -\infty$ as $s = \infty$. On the intermediate region $s \in [-1,1], t \in [0,1]$ on a neighborhood of x the section corresponds to a map to M. An argument using Lemmas 4.4 and 4.15 shows that u takes values in $U_\epsilon(x)$ everywhere. In this neighborhood L^0, L^1 may be assumed to be exact. Thus case u has zero area and, by the vanishing of the curvature in this region, must be horizontal. $\qquad\square$

Remark 5.6. (Energy gap for Floer trajectories) Let C, L^0, L^1 be as in the previous two Lemmas. We claim that there exists δ sufficiently large such that any non-constant Floer trajectory for $(L^0, C^t, C, \tau_C^{-1} L^1)$ or $(L^0, \tau_C^{-1} L^1)$ has symplectic area at least ϵ. We consider only the case of trajectories $u : \mathbb{R} \times [0,1] \to M$ for $(L^0, \tau_C^{-1} L^1)$; the case of trajectories for $(L^0, C^t, C, \tau_C^{-1} L^1)$ is similar. Choose an

open neighborhood U of m disjoint from L^1. Each component of $\tau_C^{-1}L^1 \cap U$ converges to $p^{-1}(p(C \cap L^1)) \cap U$ as $\delta \to \infty$ as smooth submanifolds. The discussion above, again using Lemma 4.15, shows that if u has sufficiently small area $A(u)$ then the image $u(\mathbb{R} \times [0,1])$ is contained in a small neighborhood of either some intersection point $m \in L^0 \cap L^1$ or an intersection point in $m \in L^0 \cap \tau_C^{-1}L^1$. Since each component of $\tau_C^{-1}L^1 \cap U$ contains at most one intersection point with m and the image of the $\mathbb{R} \times \{1\}$ under u is connected, this implies that any index one trajectory u connects an intersection point m to itself and is homotopic to the trivial trajectory. Hence u has vanishing area.

5.2.3. *Isomorphism with the mapping cone.* Every mapping cone of cochain complexes gives rise to an exact triangle. To construct an exact triangle it suffices to prove an isomorphism of a third complex with a mapping cone. So to prove Theorem 1.3 in the unquilted case (of simple Lagrangians L^0, L^1) it suffices to show the following:

Theorem 5.7. (Isomorphism with the mapping cone) *Let $L^0, L^1, \epsilon, \tau_C$ as in Theorem 1.3. Then the map $C\Phi_2$ induces an isomorphism of $CF(L^0, \tau_C^{-1}L^1)$ with the mapping cone on $C\Phi_1$.*

Before we give the proof of the theorem we recall a bit of homological algebra, explained for example in [14].

Remark 5.8. (a) (Mapping cone) If $C_j = (C_j, \partial_j), j = 0, 1$ are cochain complexes and $f : C_0 \to C_1$ is a cochain map then the mapping cone on f is the complex

$$\text{Cone}(f) := C_0[1] \oplus C_1, \quad \partial(c_0, c_1) = (-\partial_0 c_0, \partial_1 c_1 + f(c_0)).$$

(b) (Quasiisomorphisms from mapping cones) A cochain map from $\text{Cone}(f)$ to a complex C_2 is equivalent to pair (k, h) consisting of a cochain map $k : C_1 \to C_2$ together with a cochain homotopy

$$h : C_0 \to C_2, \quad k \circ f = h\partial_0 + \partial_2 h.$$

Such a map induces a quasi-isomorphism if and only if $\text{Cone}(k[1] \oplus h)$ is acyclic.

We will need the following criterion for a cochain map (k, h) as in Remark 5.8 (b) to induce a quasi-isomorphism, similar to that in Seidel [42] and Perutz [36, Lemma 5.4]. By an $\mathbb{R}$-*graded* Λ-*cochain complex* we mean a Λ-linear cochain complex equipped with an $\mathbb{R}$-grading so that multiplication by q^α shifts the grading by α.

Lemma 5.9. (Double mapping cone lemma) *Suppose that C_0, C_1 and C_2 are free, finitely-generated $\mathbb{R}$-graded Λ-cochain complexes with differentials $\delta_0, \delta_1, \delta_2$. Suppose that*

$$C_0 \xrightarrow{f} C_1 \xrightarrow{k} C_2$$

is a sequence of cochain maps (not necessarily exact at the middle term) and $h : C_0 \to C_2$ a nullhomotopy of $k \circ f$. Assume

(a) *The differentials $\delta_0, \delta_1, \delta_2$ each have positive order while h has non-negative order.*

(b) *We have $f = f_0 + f_1$, $k = k_0 + k_1$, where f_0, k_0 have order zero while f_1, k_1 have positive order.*

(c) *The leading order terms f_0, k_0 give a short exact sequence of abelian groups (not necessarily cochain complexes)*

$$0 \to C_0 \xrightarrow{f_0} C_1 \xrightarrow{k_0} C_2 \to 0$$

Then the induced map $(h, k) : \mathrm{Cone}(f) \to C_2$ is a quasiisomorphism.

Proof. The proof is similar to that in Perutz [36, Lemma 5.4]. The leading order differential in $C := \mathrm{Cone}(\mathrm{Cone}(f), C_2)$ is acyclic by standard homological algebra: Given (c_0, c_1, c_2) with leading order coboundary $(0, f_0(c_0), k_0(c_1) + h_0(c_0)) = 0$, we have $c_0 = 0$ since f_0 is injective, and so $k_0(c_1) = 0$. Hence $c_1 = f_0(b_0)$ for some $b_0 \in C_0$. Now $h_0(b_0) = -b_2$ for some $b_2 \in C_2$ and $c_2 + b_2 = k_0(b_1)$ for some $b_1 \in C_1$. So the coboundary of (b_0, b_1, b_2) is $(0, c_1, c_2 + b_2 - b_2) = (c_0, c_1, c_2)$ as desired. Since all maps f, h, k have finitely many terms, there exists an $\epsilon > 0$ such that, if $C^{\geq n} \subset C$ is the sub-complex of terms with order in $[n\epsilon, \infty)$, the first page of the associated spectral sequence has vanishing cohomology. It follows that C is itself acyclic. $\square$

Proof of Theorem 5.7. We apply Lemma 5.9 to the Floer complexes

(49)
$$\begin{aligned}
C_0 &= CF(L^0, C^t, C, \tau_C^{-1}L^1; \Lambda) \\
C_1 &= CF(L^0, \tau_C^{-1}L^1; \Lambda) \\
C_2 &= CF(L^0, L^1, \Lambda)
\end{aligned}$$

Let k denote the cochain level map $C\Phi_{1,\Lambda}$ defined by the quilted surface in Figure 6. For any $x \in \mathcal{I}(L^0, C^t, C, \tau_C^{-1}L^1)$, $z \in \mathcal{I}(L^0, L^1)$ consider the parametrized moduli space $\widetilde{\mathcal{M}}(x, z)$ for the one-parameter family of deformations $(\underline{E}_\rho, \underline{F}_\rho)$ (with ρ representing the length of the neck) connecting the pair obtained by gluing $(\underline{E}_1, \underline{F}_1)$ and (E_2, F_2) along a strip-like end to the nodal surface equipped with bundles $(\underline{E}, \underline{F}), (\underline{E}_{C,r}, \underline{F}_{C,r})$, as shown in Figure 8. An element of $\widetilde{\mathcal{M}}(x, z)$ consists of a quilted surface $\underline{S}^\rho$ in the family, together with a pseudoholomorphic quilt u^ρ with the given boundary and seam conditions.

As in the proof of Corollary 4.31, for r sufficiently small and ρ sufficiently large the parametrized moduli space is empty. The boundary of $\widetilde{\mathcal{M}}(x, z)_1$ is therefore

$$(50) \quad \partial\widetilde{\mathcal{M}}(x, z)_1 = \bigcup_y (\mathcal{M}_0(x, y)_0 \times \mathcal{M}_1(y, z)_0) \cup \bigcup_y \left(\widetilde{\mathcal{M}}(x, y)_0 \times \mathcal{M}(y, z)_0\right)$$
$$\cup \bigcup_y \left(\widetilde{\mathcal{M}}(y, z)_0 \times \mathcal{M}(x, y)_0\right);$$

where

- the first union consists of pairs of pseudoholomorphic sections of $\underline{E}_1$ and $\underline{E}_2$, and
- the second two unions correspond to bubbling off Floer trajectories $[u] \in \mathcal{M}(y, z)_0$ in M or a Floer trajectory $[u] \in \mathcal{M}(x, y)_0$ in $M \times B \times M$.

Define a map

$$h : CF(L^0, C^t, C, \tau_C^{-1} L^1; \Lambda) \to CF(L^0, L^1; \Lambda)$$

by

$$h(\langle x \rangle) = \sum_{[u] \in \widetilde{\mathcal{M}}(x,y)_0} o(u) q^{A(u)} \langle y \rangle$$

where $o(u) = \pm 1$ are the orientations constructed in [50]. The description of the boundary components of $\widetilde{\mathcal{M}}(x,z)_1$ in (50) gives the relation $\partial h + h \partial = k \circ f$. By Remark 5.8 the pair (k, h) define a morphism $\mathrm{Cone}(f) \to C_2$. We claim that the mapping cone

(51) $\quad \mathrm{Cone}(k, h) = C_0[2] \oplus C_1[1] \oplus C_2,$

$$\partial(c_0, c_1, c_2) = (\partial_0 c_0, \partial_1 c_1 + f(c_0), \partial_2 c_2 + k(c_1) + h(c_0))$$

is acyclic. Theorem 5.5 shows that for Dehn twists satisfying the conditions in the Theorem, the differential ∂ splits into the sum of an operator ∂_0 whose q-exponents lie in $[0, \epsilon/2)$ and a term $\partial - \partial_0$ whose q-exponents lie in (ϵ, ∞). Because of the bijection (42) there exists an $\mathbb{R}$-grading on C_0, C_1, C_2 so that the contributions to ∂ separates into an acyclic leading order part ∂_0 with $\mathbb{R}$-degree zero and a remaining part with $\mathbb{R}$-degree at least $\epsilon/2$. By Lemma 5.4, there exists a null-homotopy of $k \circ f$ to the identity. By Remark 5.6, for δ sufficiently small the conditions of Lemma 5.9 hold. The result for Λ coefficients follows. Note that here in contrast to the case in Perutz [36] the differentials have finitely many terms, so the formal completion is not necessary. The result for $q = 1$ follows by specialization as in Remark 4.3. $\qquad\square$

The quilted version of Theorem 1.3, where $\underline{L}^0, \underline{L}^1$ are generalized Lagrangian branes, is proved similarly, but replacing the boundary labelled L^0, L^1 with collections of strips corresponding to the symplectic manifolds appearing in the generalized Lagrangian branes $\underline{L}^0, \underline{L}^1$. The details are left to the reader.

Remark 5.10. Under strong monotonicity assumptions as in Definition 4.28, the statement of Theorem 1.3 also holds in the case of codimension one coisotropics. This case is proved similarly but now using the cancellation discussed in Remark 4.30.

5.3. **Minimal Maslov two case.** In general, Lagrangian Floer cohomology is defined only the case that certain holomorphic disk counts vanish. In the case that one of the Lagrangians has minimal Maslov number two, the relevant disk count is that of Maslov index two holomorphic disks. First we recall some basics of the derived category of matrix factorizations from [49].

Definition 5.11.   (a) (Category of matrix factorizations) For any $w \in \mathbb{Z}$, let $\mathrm{Fact}(w)$ denote the category of factorizations of $w \, \mathrm{Id}$.
   (i) The objects of $\mathrm{Fact}(w)$ consist of pairs (C, ∂), where

$$C = C = C^0 \oplus C^1$$

is a $\mathbb{Z}_2$-graded free abelian group and ∂ is a group homomorphism squaring to a multiple of the identity:

$$\partial : C^\bullet \to C^{\bullet+1}, \quad \partial^2 = w\,\mathrm{Id}\,.$$

(ii) For objects $(C, \partial), (C', \partial')$, the space of morphisms

$$\mathrm{Hom}_{\mathrm{Fact}}((C, \partial), (C', \partial'))$$

is the space of grading preserving maps intertwining the "differentials"

$$f : C^\bullet \to (C')^\bullet, \quad f\partial = \partial' f.$$

(b) (Cohomology) For any matrix factorization (C, ∂) let $H((C, \partial) \otimes_{\mathbb{Z}} \mathbb{Z}_w)$ denote the cohomology of the differential $\partial \otimes_{\mathbb{Z}} \mathrm{Id} : C \otimes_{\mathbb{Z}} \mathbb{Z}_w \to C \otimes_{\mathbb{Z}} \mathbb{Z}_w$ obtained from ∂ by tensoring with $\mathbb{Z}_w$. Any morphism in $\mathrm{Fact}(w)$ defines a homomorphism of the corresponding cohomology groups. The *cohomology with coefficients* functor has target the category Ab of $\mathbb{Z}_2$-graded abelian groups,

$$\mathrm{Fact}(w) \to \mathrm{Ab}, \quad (C, \partial) \mapsto H((C, \partial) \otimes_{\mathbb{Z}} \mathbb{Z}_w).$$

We recall the definition of the Maslov index two disk count studied in Oh [33]. Let M be a symplectic background and $L \subset M$ be a compact Lagrangian submanifold with minimal Maslov number equal to 2. For any $\ell \in L$, consider the moduli space $\mathcal{M}_2^1(L, J, \ell)$ of J-holomorphic disks $u : (D, \partial D) \to (M, L)$ with Maslov number 2, mapping a point $1 \in \partial D$ to ℓ, modulo automorphisms preserving 1. By results of Kwon-Oh [21] and Lazzarini [23], for J in a comeager subset $\mathcal{J}^{\mathrm{reg}}(M, L) \subset \mathcal{J}(M)$ the moduli space $\mathcal{M}_2^1(L, J, \ell)$ is a finite set. Suppose L is equipped with a relative spin structure. By [13], see also [50], this structure induces an orientation on the moduli space $\mathcal{M}_2^1(L, J, \ell)$. Let

$$o : \mathcal{M}_2^1(L, J, \ell) \to \{\pm 1\}$$

denote the map comparing the given orientation to the canonical orientation of a point.

Definition 5.12. (Disk invariant of a Lagrangian) Let M be a monotone symplectic background and $L \subset M$ a compact monotone Lagrangian brane with minimal Maslov number equal to 2. The *disk invariant* of L is the sum

$$w(L) = \sum_{[u] \in \mathcal{M}_2^1(L, J, \ell)} o(u) q^{A(u)} \in \Lambda.$$

The element $w(L)$ is independent of $J \in \mathcal{J}^{\mathrm{reg}}(M, L)$ and $\ell \in L$.

Denote by $\mathcal{J}_t(M, L^0, L^1) \subset \mathcal{J}_t(M)$ the subset of t-dependent almost complex structures whose restriction to a fixed small neighborhood of $t = 0$ resp. $t = 1$ lies in $\mathcal{J}^{\mathrm{reg}}(M, L^0)$ resp. $\mathcal{J}^{\mathrm{reg}}(M, L^1)$.

Proposition 5.13. (Floer cohomology) *Let L^0, L^1 compact Lagrangian branes in M. There exists a comeager subset $\mathcal{J}_t^{\mathrm{reg}}(M, \omega, L^0, L^1) \subset \mathcal{J}_t(M, \omega, L^0, L^1)$ such that*

(a) $\partial^2 = (w(L^0) - w(L^1))\,\mathrm{Id}$.
(b) $(CF(L^0, L^1; \Lambda), \partial)$ *is independent of the choice of J, H up to cochain homotopy.*

If $L^k \subset M, k = 0, 1$ are monotone Lagrangian branes with the same disk invariant, then the exact triangle in Theorem 1.3 holds, with the same proof. More generally, the disk invariant for a generalized Lagrangian brane $\underline{L} = (L_1, \ldots, L_k)$ is the sum of the disk invariants for the components $L_1, \ldots, L_k$. Furthermore, The disk invariant for a correspondences $L_{01} \subset M_0^- \times M_1$ is the *opposite* of the disk invariant for its transpose $L_{01}^t \subset M_1^- \times M_0$, because the change in complex structure reverses orientations on Maslov index two disks [50]. With these conventions, the quilted version of Theorem 1.3 also holds provided that the disk invariants of the generalized correspondences $\underline{L}^0, \underline{L}^1$ are equal.

5.4. **Periodic Floer version.** One can also formulate a version of the exact triangle for symplectomorphisms, that is, in periodic Floer theory. In this formulation, the exact triangle relates the symplectic Floer cohomology of the Dehn twist with the Lagrangian Floer cohomology of the vanishing cycle and the identity:

Theorem 5.14. (Exact triangle in periodic quilted Floer theory) *Let M, C, B, τ_C be as in Theorem 1.3. There exists a long exact sequence of Floer cohomology groups*

$$HF(\mathrm{id}) \longrightarrow HF(\tau_C)$$

$$HF(C^t, C)[\dim(B)].$$

The ideas are similar to the case of Lagrangian Floer cohomology in Theorem 1.3 and we only sketch the proof. The map $HF(\mathrm{id}) \to HF(\tau_C)$ is the relative invariant corresponding to a Lefschetz-Bott fibration over the cylinder in Figure 9 below. In the Figure the outer and inner boundary represent cylindrical ends, where the former has monodromy τ_C. The map $HF(C^t, C) \to HF(\mathrm{id})$ is the relative invariant
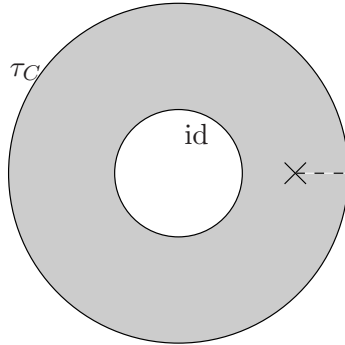

FIGURE 9. Lefschetz-Bott fibration defining $HF(\mathrm{id}) \to HF(\tau_C)$

associated to the quilted cylinder in Figure 10. In the Figure, the outer boundary represents a cylindrical end while the inner boundary represents a quilted cylindrical

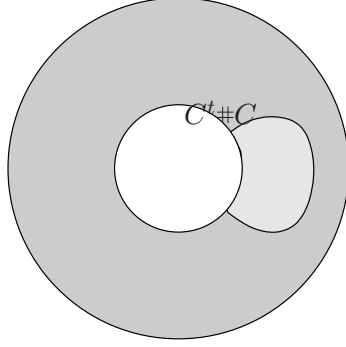end with seams labelled C^t, C. The proof is similar to that of Theorem 1.3. Namely,

FIGURE 10. Quilted surface defining $HF(C^t, C)[\dim(B)] \to HF(\mathrm{id})$

we have identifications

(52) $\qquad HF(\mathrm{id}) \cong HF(L^0, L^1), \quad HF(\tau_C^{-1}) = HF(L^0, (\tau_C^{-1} \times 1)L^1)$

where $L^0 = L^1 = \Delta$ is the diagonal in M^2. There is a natural bijection

$$\mathcal{I}(L^0, (\tau_C^{-1} \times 1)L^1) \to \mathcal{I}(L^0, L^1) \cup \mathcal{I}(L^0, C^t, C, L^1);$$

this amounts to repeating the argument of Proposition 5.2. The same filtration arguments as before are used to construct an exact triangle of *quilted* Floer cohomology groups

$$HF(L^0, \tau_C^{-1}L^1) \longrightarrow HF(L^0, L^1)$$

$$HF(L^0, C^t, C, L^1)[\dim(B)].$$

Now functoriality of quilted Floer cohomology groups under the symplectomorphism τ_C gives an exact triangle

$$HF(L^0, L^1) \longrightarrow HF(L^0, \tau_C L^1)$$

$$HF(L^0, C^t, C, L^1)[\dim(B)].$$

Using the identifications in (52) this proves Theorem 5.14. With integer coefficients, there is a complication caused by the fact that the diagonal correspondence Δ does not have a canonical relative spin structure. However, the proof goes through by treating the diagonal as the "empty correspondence", see Section 7.2 below.

6. Applications to surgery exact triangles

In this section we apply the exact triangle to obtain versions of the Floer [11], Khovanov [18], and Khovanov-Rozansky [20] exact triangles. We have already established in Section 3 that Dehn twists of surfaces induce fibered Dehn twists of moduli spaces.

6.1. **Exact triangle for three-bordisms.** In [48], we introduced a category-valued-field theory associated to certain connected, decorated surfaces and bordisms. We use freely the notation and terminology from [48]:

Definition 6.1. (a) (Decorated surface) A decorated surface is a compact connected oriented surface X equipped with a line bundle with connection $D \to X$. The *degree* of D is the integer $d = (c_1(D), X)$.
  (b) (Decorated bordisms) A *bordism* from X_- to X_+ is a compact, oriented 3-manifold Y with boundary equipped with an orientation-preserving diffeomorphism
$$\phi : \partial Y \to \overline{X}_- \sqcup X_+.$$
  Here $\overline{X}_-$ denotes the manifold X_- equipped with the opposite orientation. We often omit ϕ from the notation and assume that X_-, X_+ are boundary components of Y. A *decorated bordism* of degree d is a compact connected oriented bordism Y between decorated surfaces $(X_\pm, D_\pm)$ equipped with a line-bundle-with connection $D \to Y$ such that $D|X_\pm = D_\pm$. Given a decorated bordism (Y, D) between $(X_\pm, D_\pm)$ one can obtain another decorated bordism by tensoring D with any line bundle that is trivial on the boundary of Y.
  (c) (Moduli spaces for decorated surfaces) Suppose that X is a decorated surface with line bundle D. Let $M(X, D)$ denote the moduli space of constant curvature rank 2 bundles on X with fixed determinant. The space $M(X, D)$ is a compact, monotone symplectic manifold with monotonicity constant 1/4 with a unique Maslov cover of any even order. In [48] we describe how to equip $M(X, D)$ with the background classes so that $M(X, D)$ is equipped with the structure of a symplectic background.
  (d) (Correspondences for decorated bordisms) Suppose that Y is a decorated surface with line bundle D between decorated surfaces (X_-, D_-) and (X_+, D_+). Let
$$L(Y, D) \subset M(X_-, D_-)^- \times M(X_+, D_+)$$
  denote the set of isomorphism classes of constant central curvature bundles with fixed determinant that extend over the interior of Y. If Y is an elementary bordism (that is, admits a Morse function with a single critical point) then $L(Y, D)$ is a smooth Lagrangian correspondence. In [48] we describe how to equip $L(Y, D)$ with relative spin structures and gradings, so that $L(Y, D)$ has the structure of an admissible Lagrangian brane.

For the purposes of the exact triangle, we will need an alternative description as flat bundles with fixed holonomy around an additional marking.

Definition 6.2.   (a) (Marked surfaces) For any integer $n \geq 0$, an n *marked surface* is a compact oriented surface X equipped with a tuple $\underline{x} = (x_1, \ldots, x_n)$ of distinct points on X and a labelling $\underline{\mu} = (\mu_1, \ldots, \mu_n)$. In this section we take labels all equal to $1/2$ corresponding to the central element $-I$ of $\mathfrak{A}$.

  (b) (Moduli spaces for marked surfaces) Denote by $M(X, \underline{\mu})$ the moduli space of flat bundles on $X - \underline{x}$ with holonomy $-I$ around each marking x_i; we consider more general holonomies in the next subsection. The moduli space $M(X, \underline{\mu})$ may be identified with the moduli space $M(X, D)$ where D is a line bundle of degree n, by a construction described in Atiyah-Bott [5] given by twisting by a fixed central connection.

  (c) (Marked bordisms) A *marked bordism* from $(X_-, \underline{x}_-)$ to $(X_+, \underline{x}_+)$ is a compact, oriented bordism Y equipped with a tangle (compact oriented one-dimension submanifold transverse to the boundary) $K \subset Y$ such that $K \cap X_\pm = \underline{x}_\pm$.

  (d) (Moduli spaces for marked bordisms) Denote by

$$L(Y, K) \subset M(X_-, \underline{\mu}_-)^- \times M(X_+, \underline{\mu}_+)$$

the moduli space of bundles that extend over $Y - K$ with holonomy $-I$ around K. The identification $M(X_\pm, \underline{\mu}_\pm) \to M(X_\pm, D_\pm)$ induces a homeomorphism $M(Y, K) \to M(Y, D)$ where $D \to Y$ is a line bundle whose first Chern class is dual to the tangle K. Thus, in particular, the addition to K of a circle component K' corresponds to twisting the determinant line bundle D by a line bundle whose first Chern class is dual to the homology class of K'.

The following proposition relates the moduli spaces of bundles with fixed holonomy around an embedded circle with the Lagrangian correspondences associated to elementary bordisms.

Proposition 6.3. (Correspondences for elementary bordisms)

  (a) *Let Y be a bordism from X_- to X_+ containing a single critical point of index 1 and a trivial tangle K (that is, a union of intervals connecting $\underline{x}_-$ to $\underline{x}_+$) and $C \subset X_+$ is the attaching cycle. The Lagrangian $L(Y, K)$ is diffeomorphic via the projection to $M(X_+, \underline{\mu}_+)$ to the subset of connections on $X_+ - \underline{x}_+$ with holonomy along C equal to I.*

  (b) *Let Y be a decorated bordism from X_- to X_+ containing a single critical point of index 1, $C \subset X_+$ the attaching cycle, K_0 a trivial bordism connecting $\underline{x}_-$ to $\underline{x}_+$ and $K_1 \subset Y$ the unstable manifold of the critical point. The Lagrangian $L(Y, K_0 \cup K_1)$ is diffeomorphic to the subset of flat bundles on $X_+ - \underline{x}_+$ with holonomy along C equal to $-I$.*

Proof. By Seifert-van Kampen, $\pi_1(Y - K)$ is the quotient of $\pi_1(X_+ - K)$ by the ideal generated by the element $[C]$ obtained from C by joining by a path to the base point. Hence in the first case, $L(Y, K)$ is diffeomorphic to the submanifold of $M(X, \underline{\mu}_+)$ obtained by setting the holonomy along C equal to the identity. For the second case, the gradient flow the Morse function defines a deformation retract of $Y - K_0 - K_1$ to $X_+ - \underline{x}_+$. Homotopy invariance implies that $\pi_1(Y - K_0 - K_1)$ is

isomorphic to $\pi_1(X_+ - \underline{x}_+)$. Since C is a loop around K_1, the holonomy around K_1 is equal to the holonomy along C, hence the claim. $\square$

In order to obtain smooth Lagrangian correspondences, we break the given bordism into elementary bordisms.

Definition 6.4.   (a) (Cerf decompositions) A *Cerf decomposition* of a bordism Y is a decomposition of Y into elementary bordisms $Y_1, \ldots, Y_K$, that is, bordisms admitting a Morse function with at most one critical point. Associated to any Cerf decomposition

$$Y = Y_1 \cup_{X_1} \ldots \cup_{X_{k-1}} Y_k$$

and a decoration on Y is a *generalized Lagrangian correspondence*

$$\underline{L}(Y) = (L(Y_1), \ldots, L(Y_k)).$$

(b) The generalized correspondence $\underline{L}(Y)$ may be equipped with a relative spin structure via its structure as a fibration over the moduli space of the incoming or outgoing surface. Thus $\underline{L}(Y)$ gives rise to a functor of generalized Fukaya categories

$$\Phi(Y) : \mathrm{Fuk}^{\#}(M(X_-)) \to \mathrm{Fuk}^{\#}(M(X_+)).$$

This functor is independent of the choice of Cerf decomposition [48].

(c) Given admissible Lagrangian branes $L^{\pm} \subset M(X_{\pm})$, define

$$
\begin{aligned}
HF(Y; L^-, L^+) &:= H(\mathrm{Hom}_{\mathrm{Fuk}^{\#}(M(X_+))}(\Phi(Y)L^-, L^+)) \\
&:= HF(L^-, \underline{L}(Y), L^+)
\end{aligned}
$$

where the second equality is by definition of the Fukaya category. Since we have ignored absolute gradings and the minimal Chern number is two this is a relatively $\mathbb{Z}_4$-graded group depending only on the equivalence class of the decorated bordism Y.

We prove the following surgery exact triangle for the invariants $HF(Y; L^-, L^+)$.

Definition 6.5.   (a) (Knots) A *knot* in a bordism Y is an embedded, connected 1-manifold $K \subset Y$ disjoint from the boundary.

(b) (Knot framings) A *framing* of a knot $K \subset Y$ is a non-vanishing section of its normal bundle up to homotopy. Given a framed knot, the other framings are obtained by twisting by representations of $\pi_1(K) \cong \mathbb{Z}$ into $SO(2)$, and so are indexed by $\mathbb{Z}$.

(c) (Knot surgeries) For $\lambda \in \mathbb{Z}$ the *λ-surgery* $Y_{\lambda,K}$ of Y along K is obtained by removing a tubular neighborhood of K and gluing in a solid torus $D^2 \times S^1$ so that the *meridian* $\partial D^2 \times \{\mathrm{pt}\}$ is glued along the curve given by the framing of the knot corresponding to λ. Denote by K_λ the knot in $Y_{\lambda,K}$ corresponding to a longitude in $\partial D^2 \times S^1$. Thus K_λ intersects the meridian transversally once.

Remark 6.6. (Knot surgeries in terms of decompositions into elementary bordisms) The three-manifolds $Y_{0,K}, Y_{-1,K}$ obtained by a 0 resp. -1-knot surgery have decompositions into elementary bordisms given as follows. Suppose that Y is decomposed into elementary bordisms $Y_1, \ldots, Y_l$, so that K is contained in the boundary $(\partial Y_i)_+ = (\partial Y_{i+1})_-$ and the framing is the direction normal to the boundary. Gluing in $D^2 \times S^1$ produces two new critical points. The first critical point has stable manifold with unit sphere equivalent to K and the second has unstable manifold with unit sphere equivalent to K. Thus

    (a) The zero-surgery $Y_{0,K}$ has a decomposition into elementary bordisms with two additional pieces, $Y_\cup, Y_\cap$ inserted between Y_i and Y_{i+1}. The knot $K_0 \subset Y_{0,K}$ is divided into the two additional pieces $K_0 \cap Y_\cup$ and $K_0 \cap Y_\cap$. The correspondence $L(Y_\cup, Y_\cup \cap K_0) \circ L(Y_\cap, Y_\cap \cap K_0)$ is the correspondence associated to the moduli space of bundles $L(Y_\cup) \circ L(Y_\cap)$ on the decorated surface $Y_\cup \cup Y_\cap$ with the *shifted line bundle* as in Proposition 6.3.

    (b) The -1 surgery $Y_{-1,K}$ has decomposition into simple bordisms $Y_1, \ldots, Y_l$ but where the identification $(\partial Y_i)_+ \to (\partial Y_{i+1})_-$ is the Dehn twist along K.

Lemma 6.7. (Existence of Cerf decompositions compatible with a knot) *For any framed knot $K \subset Y$, there exists a Cerf decomposition $Y = Y_1 \cup \ldots \cup Y_l$ so that K is contained in the boundary $(\partial Y_i)_+ = (\partial Y_{i+1})_-$ for some $i = 1, \ldots, l$ and the framing is the direction normal to the boundary.*

Proof. Choose a Morse function $f : Y \to \mathbb{R}$ such that f is constant on K and the framing is given by the gradient of f at K. The level set $f^{-1}(\lambda)$ containing K can be made connected by adding 1-handles in Y, so that $f^{-1}(\lambda)$ becomes a connected surface containing K separating the boundary components of Y. By taking a self-indexing Morse function on $f^{-1}((\pm\infty, \lambda])$ one obtains that f is Morse and has connected fibers. $\square$

Theorem 6.8. (Exact triangle for knot surgery) *Let Y be a decorated bordism from X_- to X_+, let $K \subset Y$ be a framed knot contained in the interior of Y, and let $Y_{-1,K}, Y_{0,K}$ denote the -1 and 0-surgeries on K. Let L^-, L^+ be admissible Lagrangian branes in $M(X_\pm)$. There is a long exact sequence of (relatively graded) Floer cohomology groups*

$$\ldots \to HF(Y_{0,K}; L^-, L^+) \to HF(Y; L^-, L^+) \to HF(Y_{-1,K}; L^-, L^+) \to \ldots$$

where the determinant bundle on $Y_{0,K}$ has been shifted by the dual class of the knot $K_0 \subset Y_{0,K}$, or equivalently, $Y_{0,K}$ is considered as a marked bordism with knot K_0.

Proof. By combining Remark 6.6, Lemma 6.7, Theorem 3.8, the statement becomes a special case of Theorem 1.3: Let $Y = Y_- \cup_X Y_+$ be a decomposition into compression bodies with $K \subset X$ a non-separating knot and τ a corresponding Dehn twist. We have an exact sequence

(53)
$$\ldots \to HF(L(Y_-), L(Y^\cup, Y^\cup \cap K_0), L(Y^\cap, Y^\cap \cap K_0), L(Y_+)) \to HF(L(Y_-), L(Y_+))$$
$$\to HF(L(Y_-), \mathrm{graph}((\tau^{-1})^*), L(Y_+)) \to \ldots.$$

The addition of the knot K_0 is equivalent to a shift in the determinant line bundle, as explained in Remark 6.6. We thank Guillem Cazassus for pointing out the missing shift in an earlier version of the paper; see [8]. □

Remark 6.9. (Dehn twists for separating curves) We do not discuss here the exact triangle for a Dehn twist around a separating curve, because the moduli space M does not satisfy the "strong monotonicity" condition of Definition 4.28. Instead, one needs to establish positivity properties of the form $c_1(M_C) - [B_-] - [B_+]$ with respect to the *canonical* complex structure on the moduli space of parabolic bundles M_C. This seems like to hold but would take us outside the framework of this paper.

6.2. **Exact triangles for tangles.** We obtain Floer-theoretic invariants of tangles constructed in [49] exact triangles that are the same as those obtained by Khovanov [19, 18] and Khovanov-Rozansky [20]. We assume freely the terminology from [49], in particular the terminology for moduli spaces of bundles with fixed holonomy for marked surfaces. In this subsection we take $G = SU(2)$ and $\mathfrak{A} \cong [0, 1/2]$ via the identification (12).

Definition 6.10. (Functors for bordisms with tangles via flat bundles with fixed holonomy)

(a) (Correspondences for elementary tangles) Let $X_\pm$ be a compact, oriented surface with odd numbers of markings $\underline{x}_\pm$ admissible labels $\underline{\mu}_\pm$ all equal to $1/4 \in \mathfrak{A}$. Let $K \subset Y := X \times [-1, 1]$ be a tangle, that is, a bordism between marked surfaces $(X_-, \underline{\mu}_-)$ and $(X_+, \underline{\mu}_+)$. Let $M(X_\pm, \underline{\mu}_\pm)$ denote the moduli space of $SU(2)$ bundles in Definition 3.1. Let $L(K)$ denote the subset of $M(X_-, \underline{\mu}_-)^- \times M(X_+, \underline{\mu}_+)$ of bundles that extend over the interior of the bordism. Assuming that K is elementary (admits a Morse function with at most one critical point on K, and none on Y) the Lagrangian $L(K)$ has an admissible brane structure [49].

(b) (Functors for tangles) More generally, let $K \subset Y = X \times [-1, 1]$ be an arbitrary tangle, and $K = K_1 \cup \ldots \cup K_r$ a decomposition into elementary tangles. Associated to each elementary tangle K_i is a Lagrangian correspondence L_i and so a generalized Lagrangian correspondence associated to K

$$\underline{L}(K) = (L_1, \ldots, L_r).$$

Composing the functors $\Phi(L_i)$ for these correspondences gives a functor

$$\Phi(K) : \mathrm{Fuk}^\#(M(X_-, \underline{\mu}_-)) \to \mathrm{Fuk}^\#(M(X_+, \underline{\mu}_+));$$

In [49] we proved that $\Phi(K)$ is independent, up to isomorphism, of the choice of decomposition $K_1 \cup \ldots \cup K_r$.

(c) (Group valued invariants) Given objects

$$L^\pm \in \mathrm{Obj}(\mathrm{Fuk}^\#(M(X_\pm, \underline{\mu}_\pm))),$$

such that the sum of disk invariants for $L_-, L(K), L_+$ is zero, the cohomology

$$HF(K; L^-, L^+) = H(\mathrm{Hom}_{\mathrm{Fuk}^\#(M(X_+, \underline{\mu}_+))}(\Phi(K)L^-, L^+))$$

is a ($\mathbb{Z}_2$-relatively graded) invariant of K.

We prove the following surgery exact triangle for these invariants.

Definition 6.11. Given a tangle $K \subset Y$, a separating embedded surface $\Sigma \subset Y$, and a disk $D \subset \Sigma$ meeting K in two points. let $K^{\backsim}, K^{\times}$ be the tangles obtained by modifying K by a half-twist, respectively adding a cup and cap as in Figure 11.
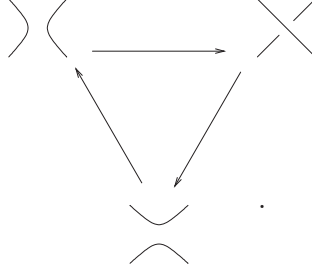

FIGURE 11. Exact triangle for a crossing change, $SU(2)$ case

Theorem 6.12. (Exact triangle for changing a crossing) *Let $K, K^{\backsim}, K^{\times}$ be tangles in a cylindrical bordism $Y = X \times [-1, 1]$ between $X_{\pm} = X \times \{\pm 1\}$ as in Definition 6.11 and $L^{\pm}$ admissible Lagrangian branes in $M(X_{\pm}, \underline{\mu}_{\pm})$ such that the sum of the disk invariants for $\underline{L}(K), L^-, L^+$ is zero. The cohomology groups for $K^{\times}, K, K^{\backsim}$ are related by an exact triangle*

$$
\begin{array}{ccc}
HF(K; L^-, L^+) & \longrightarrow & HF(K^{\times}; L^-, L^+). \\
& & \\
& HF(K^{\backsim}; L^-, L^+) &
\end{array}
$$

(54)

Proof. The crossing change depicted in Figure 11 has the following effect on the decomposition of elementary tangles. Suppose that $K = K_1 \cup \ldots \cup K_r$ is such a decomposition. Denote by

$$L_i \subset M(X_i, \underline{\mu}_i)^- \times M(X_{i+1}, \underline{\mu}_{i+1}), \ i = 1, \ldots, r$$

the corresponding Lagrangian correspondences. The tangle $K^{\times}$ is obtained by inserting a half-twist of the j and $j+1$-st markings after some elementary tangle K_i. We may assume that K_i is a cylindrical bordism (that is, admits a Morse function with no critical points) so that the cylindrical Cerf decomposition of $K^{\times}$ is obtained from that of K by replacing K_i with a half-twist. Similarly for $K^{\backsim}$, let $K_j^{\cap}, K_j^{\cup}$ denote the tangles corresponding to the cap and cup of the j-th and $j+1$-st strands. Then

$$K^{\backsim} = K_1 \cup \ldots \cup K_{i-1} \cup K_j^{\cap} \cup K_j^{\cup} \cup K_{i+1} \cup \ldots \cup K_r$$

is a cylindrical Cerf decomposition of $K^{\asymp}$. The Lagrangian correspondence for $K_j^{\cup}$ is that associated to the coisotropic submanifold

$$C_j^{\cup} = \{g_j g_{j+1} = 1\} \subset M(X_i, \underline{\mu}_i)$$

where g_i is the holonomy around the j-th marking. Similarly the correspondence for $K_j^{\cap}$ is $C_j^{\cap} = (C_j^{\cup})^t$. These correspondences are simply-connected, hence automatically monotone. Let $\tau_{C_j} \in \mathrm{Diff}(M(X_{i+1}, \underline{\mu}_{i+1}))$ denote the corresponding fibered Dehn twist. We have

$$\begin{aligned}
CF(K; L^-, L^+) &= CF(L^-, L_1, \ldots, L_r, L^+) \\
CF(K^{\asymp}; L^-, L^+) &= CF(L^-, L_1, \ldots, L_i, C_j^{\cap}, C_j^{\cup}, L_{i+1}, \ldots, L_r, L^+) \\
CF(K^{\times}; L^-, L^+) &= CF(L^-, L_1, \ldots, L_{i-1}, (\tau_{C_j} \times 1)L_i, L_{i+1}, \ldots, L_r, L^+).
\end{aligned}$$

Theorem 6.12 now follows from Theorems 1.3 and 3.11. $\square$

More generally, in higher rank invariants we obtain an exact triangle for the *Khovanov-Rozansky modification*.

Definition 6.13. Let $G = SU(r)$ for some integer $r \geq 2$.
  (a) (Admissible labels) An *admissible label* is a projection of the barycenter of $\mathfrak{A}$ onto some face. In the absence of reducibles, given a marked surface $(X, \underline{x})$ with admissible labels $\underline{\mu}$ the moduli space $M(X, \mu)$ of bundles with fixed holonomy around the markings $\underline{x}$ in the conjugacy classes associated to $\underline{\mu}$ is a smooth, compact, monotone symplectic manifold.
  (b) (Correspondences for admissible graphs) Let

$$K \subset Y := X \times [-1, 1]$$

be a trivalent graph with admissible labels, where each trivalent vertex is of the form described in Definition 3.12. This means that K is made up of a finite number of *vertices* that are points in Y, and *edges* that are embedded, compact one-manifold with boundary. The endpoints of the edges are either vertices or points on the boundary of Y. Trivalence means that each vertex is required to lie in the boundary of exactly three edges. Each edge is labelled either by ν_k^1 or ν_k^2, so that at any vertex the labels are $\nu_k^1, \nu_k^1, \nu_k^2$. See Figure 12 where the squiggly edge represents a vertex labelled ν_k^2.
  (c) (Decomposition into elementary graphs) Let

$$K = K_1 \cup \ldots \cup K_e$$

be a decomposition into *elementary graphs* admitting cylindrical Morse functions with at most one critical point or vertex. Each elementary graph K_i defines a smooth Lagrangian correspondence with admissible brane structure L_i consisting of bundles that extend over the interior, see [49], and so a generalized Lagrangian correspondence $\underline{L} = (L_1, \ldots, L_e)$. The functor

$$\Phi(K) : \mathrm{Fuk}^{\#}(M(X_-, \underline{\mu}_-)) \to \mathrm{Fuk}^{\#}(M(X_+, \underline{\mu}_+))$$

obtained by composing the functors $\Phi(L(K_i))$ is independent of the choice of decomposition into elementary graphs [49], up to A_∞ homotopy.

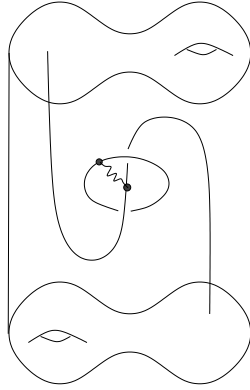FIGURE 12. Trivalent graph in a bordism

(d) (Khovanov-Rozansky modification of a graph) Suppose $K \subset X \times [-1, 1]$ is a trivalent graph with admissible labels. Let $(K_1, \ldots, K_e)$ be a cylindrical Cerf decomposition of K and $K \cap (X \times \{b_i\})$ a slice such that two points have the same label ν_k^1 from 3.12. We obtain a new trivalent graphs $K^\times, K^\times$ by inserting a half-twist, respectively inserting two new vertices as shown in Figure 13. Here the intermediate edge represented by a squiggle is labelled ν_k^2 from 3.12.
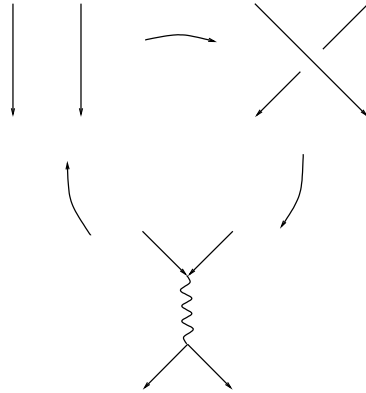
FIGURE 13. Exact triangle for a crossing change, $G = SU(r)$ case

By Theorems 1.3 and 3.13,

Theorem 6.14. (Exact triangle for a Khovanov-Rozansky modification) *Suppose that $G = SU(r)$, $(X, \underline{\mu})$ is a marked surface with $\mu_i = \mu_j = \nu_k^1$, with $\nu_k^1 = \frac{1}{2}(\omega_k + \omega_{k+1})$. Let $\nu_k^2 = \frac{1}{2}(\omega_{k+2} + \omega_k)$ and suppose that $K, K^\times, K^\times$ are as in Definition 6.13, and $L^\pm$ are Lagrangian branes such that the sum of the disk invariants for $\underline{L}(K), L^-, L^+$ is zero. There exists an exact triangle*

$$HF(K;L^-,L^+) \longrightarrow HF(K^\times;L^-,L^+)$$

(55) $$HF(K^\times;L^-,L^+).$$

Note that this generalizes the $SU(2)$ exact triangle since if $k = 2$ then $\omega_0 = \omega_2 = 0$. The exact triangle of Khovanov-Rozansky [20] has a similar form. The theories for the other standard markings will not in general have surgery exact triangles of this form, since the corresponding symplectomorphisms are not, in general, Dehn twists. It would be interesting to understand whether there is a replacement for the surgery exact triangle in these more general cases.

7. Fukaya versions of the exact triangle

In [28] the authors constructed A_∞ functors for Lagrangian correspondences between Fukaya categories. The gluing results necessary for the construction of the A_∞ functors for Lagrangian correspondences are proved in [29]. Applied to the Lagrangian correspondences arising from moduli spaces of flat bundles one obtains a (partial) A_∞ -category-valued field theory. We now explain the Fukaya-categorical versions of the exact triangle for fibered Dehn twist.

7.1. **Open Fukaya-categorical version.** Let M be a symplectic background. We take $\mathrm{Fuk}(M)$ to be the A_∞ category whose objects are admissible monotone branes with torsion fundamental group and minimal Maslov number at least three. That is, we disallow the minimal Maslov number two case. Morphisms are Floer cochain groups and composition maps count (perturbed) holomorphic polygons with boundary on the given Lagrangians. In [28] we defined a similar category $\mathrm{Fuk}^\#(M)$ whose objects are generalized Lagrangian submanifolds. Let $D^b\mathrm{Fuk}^\#(M)$ denote its bounded derived category, as defined by Kontsevich, see [43]. Given any Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$ with admissible brane structure, [28] constructs an A_∞ functor

$$\Phi(L_{01}) : \mathrm{Fuk}^\#(M_0) \to \mathrm{Fuk}^\#(M_1).$$

The functor $\Phi(L_{01})$ is defined on the level of objects by concatenating L_{01} to the sequence, and on the level of morphisms by counting holomorphic quilts. In particular, given a spherically fibered coisotropic $M \supset C \to B$ of codimension $c \geq 2$ with admissible brane structure, one obtains A_∞ functors

$$\Phi(C) : \mathrm{Fuk}^\#(M) \to \mathrm{Fuk}^\#(B), \quad \Phi(C^t) : \mathrm{Fuk}^\#(B) \to \mathrm{Fuk}^\#(M).$$

Theorem 7.1. (Exact triangle in the derived Fukaya category) *Let (M, ω) be a symplectic background, $L \subset M$ an admissible Lagrangian brane, and $\iota : C \to M$ a coisotropic submanifold of codimension at least 2 equipped with admissible brane structure spherically fibered over a symplectic manifold B. Let $\tau_C : M \to M$ denote*

a fibered Dehn twist along C. There is an exact triangle in $D^\flat \mathrm{Fuk}^\#(M)$ of the form

$$L \longrightarrow \tau_C L$$

$$\Phi(C)\Phi(C^t)L.$$

The morphisms in the triangle Theorem 7.1 are Floer cochains defined as relative invariants associated to quilted surfaces that are variations of the surfaces involved in the proof of Theorem 1.3. The first map

$$f = C\Phi_S \in CF(L, C^t, C, L)[\dim(B)] = \mathrm{Hom}(\Phi(C)\Phi(C^t)L, L)$$

is obtained by counting the elements of the moduli spaces $\mathcal{M}_1(x), x \in \mathcal{I}(L, C^t, C, L)$ associated to the quilted surface S shown in Figure 14 with Lagrangian boundary and seam conditions L, C:

$$f = \sum_{\substack{x \in \mathcal{I}(L,C^t,C,L) \\ u \in \mathcal{M}_1(x)_0}} o(u)\langle x \rangle.$$
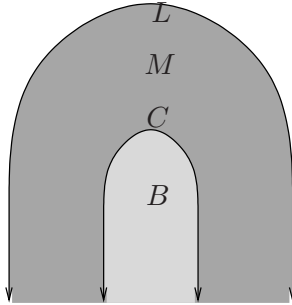
FIGURE 14. Quilted surface defining the morphism from $\Phi(C)\Phi(C^t)L$ to L

The second map in the exact triangle is the relative invariant for the standard Lefschetz-Bott fibration with a single end and Lagrangian boundary condition L in Figure 15. That is, if $\mathcal{M}_2(y)_0$ denotes the zero-dimensional component of the moduli space of pseudoholomorphic sections then the map is

(56) $$k = \sum_{\substack{y \in \mathcal{I}(\tau_C^{-1}L,L) \\ u \in \mathcal{M}_2(y)_0}} o(u)\langle y \rangle.$$

Proposition 7.2. *The morphism $k \in \mathrm{Hom}(L, \tau_C L)$ of (56) induces an isomorphism of the mapping cone $\mathrm{Cone}(f : \Phi(C)\Phi(C^t)L \to L)$ with $\tau_C L$.*

The proof of Proposition 7.2 depends on the following lemma, whose proof is left as an exercise (c.f. [38, Lemma 2.6]).

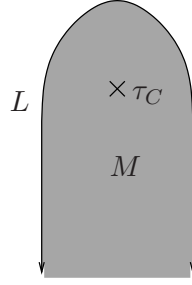FIGURE 15. Lefschetz-Bott fibration defining the morphism from $\tau_C^{-1}L$ to L

Lemma 7.3. (Sufficient condition for a morphism from a mapping cone to be an isomorphism) *Let $\mathcal{C}$ be a c-unital A_∞ category. Let X, Y, Z be objects of $\mathcal{C}$ and*

$$f \in \mathrm{Hom}^0(X, Y).$$

Any pair

$$k \in \mathrm{Hom}^0(Y, Z), \quad h \in \mathrm{Hom}^{-1}(X, Z)$$

satisfying

(57) $$\mu^1(k) = 0, \quad \mu^1(f) = 0, \quad \mu^1(h) + \mu^2(f, k) = 0$$

defines a morphism $\mathrm{Cone}(f) \to Z$ in $D^\flat(\mathcal{C})$. This is an isomorphism if for all $W \in \mathrm{Ob}(\mathcal{C})$, the complex

$$\mathrm{Hom}(W, X)[2] \oplus \mathrm{Hom}(W, Y)[1] \oplus \mathrm{Hom}(W, Z)$$

with differential

(58) $$(a, b, c) \mapsto (\mu^1(a), \mu^1(b) + \mu^2(a, f), \mu^1(c) + \mu^2(b, k) + \mu^2(a, h) + \mu^3(a, f, k))$$

is acyclic.

Proof of Proposition 7.2. Consider the relative cochain-level invariant associated to the surface in the center in Figure 16. We consider a family of deformations of this
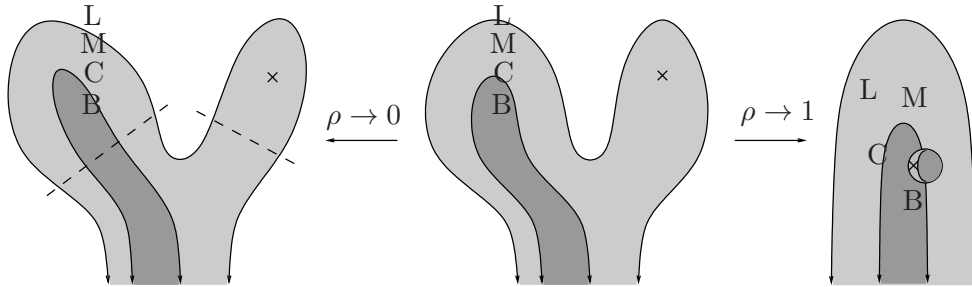

FIGURE 16. Null-homotopy for the composition $\Phi(C)\Phi(C^t)L$ to $\tau_C L$

surface depending on a parameter ρ as follows. As $\rho \to 1$, deform the glued surface so that a disk with values in $E_{C,r}$ bubbles off. By the proof of Proposition 4.31, the

relative invariant for the picture on the right corresponding to $\rho = 1$ is zero on the cochain level for r sufficiently large. As $\rho \to 0$, we pinch off a pair of pants as in the left side of Figure 16. Let

$$\widetilde{\mathcal{M}}(x) = \bigcup_{\rho \in [0,1]} \{\rho\} \times \widetilde{\mathcal{M}}^\rho(x)$$

denote the parametrized moduli space for this deformation consisting of pairs of a parameter ρ and a holomorphic quilt for the quilt corresponding to the parameter. Standard transversality arguments show that $\widetilde{\mathcal{M}}(x)$ is smooth for generic choices of perturbation data. The monotonicity conditions imply the absence of sphere and disk bubbling, hence compactness of the moduli space. The boundary of the moduli space admits a natural identification

$$\partial\widetilde{\mathcal{M}}(x)_1 \cong \bigcup_{y,z}(\mathcal{M}_1(y)_0 \times \mathcal{M}_2(z)_0 \times \mathcal{M}(x,y,z)_0) \cup \bigcup_{y}(\widetilde{\mathcal{M}}(y)_0 \times \mathcal{M}(x,y)_0).$$

Here $\mathcal{M}(x,y), \mathcal{M}(x,y,z)$ are the moduli spaces for the 2 resp. 3 marked disk, counted by the compositions μ^1, μ^2. The first part consists of the $\rho = 0$ boundary of $\widetilde{\mathcal{M}}(x)_1$, and corresponds to $\mu^2(f,k)$. The other boundary components at $\rho \in (0,1)$ are formed by splitting off Floer trajectories $v \in \mathcal{M}(x,y)$ for L, C^t, C, L. Define

$$h \in CF(L,C^t,C,L), \quad h = \sum_{(u,\rho)\in\widetilde{\mathcal{M}}(y)_0} o(u)q^{A(u)}\langle y \rangle.$$

Then by counting the ends of the one-dimensional component of the moduli space we obtain

$$0 = \sum_{\substack{x\in\mathcal{I}(L,C^t,C,L) \\ (u,\rho)\in\partial\widetilde{\mathcal{M}}(x)_1}} o(u)q^{A(u)}\langle x \rangle = \mu^2(f,k) + \mu^1(h)$$

as claimed.

Now let L^1 be another object in $D^\flat\text{Fuk}^\#(M)$, for simplicity unquilted. Acyclicity of the differential (58) is shown as follows. It suffices to prove acyclicity with L replaced with $\tau_C^{-1}L$. The terms of lowest order in q are $\mu^2(f,a)$ and $\mu^2(k,b)$. As in Section 5.2, the leading term of $\mu^2(f,a)$ corresponds to the canonical injection $\mathcal{I}(L^1,C^t,C,\tau_C^{-1}L) \to \mathcal{I}(L^1,\tau_C^{-1}L)$. On the other hand, the leading term of $\mu^2(k,b)$ corresponds to the canonical injection $\mathcal{I}(L^1,L) \to \mathcal{I}(L^1,\tau_C^{-1}L)$. As before, the lowest order terms in complex are acyclic, after a small shift in the $\mathbb{R}$-degrees of the generators. Filtering the complex by energy shows that entire complex is acyclic. An application of Lemma 7.3 completes the proof of Proposition 7.2. $\square$

Theorem 7.1 follows by taking the long exact sequence associated to the mapping cone in Proposition 7.2.

7.2. **Periodic Fukaya-categorical version.**
In this section we discuss a version of the triangle taking values in $D^\flat\text{Fuk}^\#(M,M)$, the bounded derived Fukaya category of generalized Lagrangian correspondences from M to M. Recall from [28] that

the empty correspondence $\emptyset$ from M to M considered as a sequence of length zero induces the identity

$$\Phi(\emptyset) = \mathrm{Id} : \mathrm{Fuk}^{\#}(M) \to \mathrm{Fuk}^{\#}(M)$$

on the Fukaya category; indeed, labelling each seam by the empty set has the effect of "removing the seam".

Theorem 7.4. (Exact triangle in the derived Fukaya category of correspondences) *Let (M, ω) be a symplectic background, and $i : C \to M$ a coisotropic submanifold of codimension at least two, whose null foliation $p : C \to B$ is spherically fibrating over a manifold B, equipped with an admissible brane structure. Let $\tau_C : M \to M$ denote a fibered Dehn twist along C. There is an exact triangle in $D^{\flat}\mathrm{Fuk}^{\#}(M, M)$ of the form*

$$\emptyset \longrightarrow \mathrm{graph}(\tau_C)$$

$$C^t \# C[\dim(B)].$$

Sketch of proof. The proof is the similar to that of Theorem 7.2, replacing the strip-like ends with cylindrical ends. The morphism from $C \# C^t$ to $\emptyset$ is obtained from the
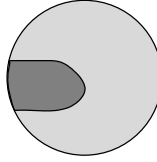

FIGURE 17. Quilt defining the morphism from $(C^t, C)[\dim(B)]$ to $\emptyset$

quilted cap $\underline{S} = (S_M, S_B)$ in Figure 17, where

- the outer circle represents a quilted cylindrical end with seams C^t, C,
- the lightly shaded patch S_M maps to M and
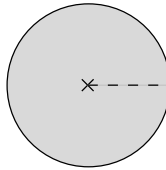- the darkly shaded patch S_B maps to B.

FIGURE 18. Quilt defining the morphisms from $\emptyset$ to $\mathrm{graph}(\tau_C)$

The morphism from $\emptyset$ to $\mathrm{graph}(\tau_C)$ is defined by the Lefschetz-Bott fibration over the cap shown in Figure 18, where the outer circle represents a cylindrical end with monodromy around the end given by τ_C. The composition of the two maps is defined

by the surface shown in Figure 19. By deforming the singularity on the surface onto a disk with boundary condition in C, one obtains a null-homotopy of the composition. This null-homotopy defines, as in Lemma 7.3, an isomorphism of the mapping cone $\mathrm{Cone}(C^t \# C[\dim(B)] \to \emptyset)$ with τ_C. Applying the functor
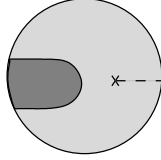

FIGURE 19. Composition of quilts

$$\mathrm{Fuk}^{\#}(M, M) \to \mathrm{Func}(\mathrm{Fuk}^{\#}(M), \mathrm{Fuk}^{\#}(M))$$

constructed in [28] this leads to an exact triangle in $\mathrm{Func}(D^{\flat}\mathrm{Fuk}^{\#}(M), D^{\flat}\mathrm{Fuk}^{\#}(M))$

$$\mathrm{id} \xrightarrow{\hspace{2cm}} D^{\flat}\Phi(\tau_C)$$

$$D^{\flat}\Phi(C) \circ D^{\flat}\Phi(C^t).$$

That is, there exists an isomorphism in $\mathrm{Func}(D^{\flat}\mathrm{Fuk}^{\#}(M), D^{\flat}\mathrm{Fuk}^{\#}(M))$,

$$D^{\flat}\Phi(\tau_C) \to \mathrm{Cone}(D^{\flat}\Phi(C) D^{\flat}\Phi(C^t) \to \mathrm{id}).$$

Applying this exact triangle to any object L of $D\mathrm{Fuk}^{\#}(M)$ leads to the exact triangle given in Theorem 7.1. $\qquad\square$

Remark 7.5. (A_{∞} results for minimal Maslov two) Similar results hold in the case case of minimal Maslov number two for the A_{∞} categories $\mathrm{Fuk}^{\#}(M, w)$ whose objects L have disk invariant $w = w(\underline{L})$, counting the number of Maslov index disks passing through a generic point in $\underline{L}$. See [28, Section 4.4] for more on the disk invariant and Fukaya category $\mathrm{Fuk}^{\#}(M, w)$.

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